Invariance for Measure Driven Dynamical Systems

Valeriano A. de Oliveira, Fernando L. Pereira, Geraldo N. Silva

Departamento de Ciência dos Computadores e Estatística
Universidade Estadual Paulista - UNESP
15054-000 - S. J. Rio Preto-SP, Brasil
gsilva@dcce.ibilce.unesp.br

Department of Engineering Electrotechnics and Computers
Faculdade de Engenharia da Universidade do Porto
Rua Dr. Roberto Frias, 4200-465, Porto, Portugal
flp@fe.up.pt

Abstract

The conventional concepts of invariance are extended in this article to include impulsive control systems represented by measure driven differential inclusions. Invariance conditions are derived and their main features are illustrated with simple examples. The solution concept plays a critical role in the extension of the conventional conditions for the impulsive control context.

Keywords: Invariance, impulsive control, measure driven differential inclusion.

1 Introduction

In this article, we provide a definition of invariance and derive invariance conditions for impulsive control problems where the dynamics are specified by measure driven. These can be regarded as an extension of the corresponding results for conventional control systems presented in [11] to the impulsive control context.

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We consider impulsive control systems of the form

\[
\begin{align*}
\begin{cases}
\frac{dx(t)}{dt} &\in F(t, x(t)) + G(t, x(t))\mu(dt), \quad t \in [0, \infty) \\
x(0) &\in C_0,
\end{cases}
\end{align*}
\]

where \( F : [0, \infty) \times \mathbb{R}^n \rightrightarrows \mathcal{P}(\mathbb{R}^n) \), \( G : [0, \infty) \times \mathbb{R}^n \rightrightarrows \mathcal{P}(\mathbb{R}^{n \times q}) \) are given multi-functions and \( \mu \in C^\ast([0, \infty); K) \), the set in the dual space of continuous functions from \([0, \infty)\) into \( \mathbb{R}^q \) with values in \( K \). The set \( K \) is the positive cone in \( \mathbb{R}^q \).

In what follows \( AC([0, \infty); \mathbb{R}^n) \) and \( BV^+([0, \infty); \mathbb{R}^n) \) mean the space of absolutely continuous \( \mathbb{R}^n \)-valued functions on \([0, \infty)\) and the vector space of \( \mathbb{R}^n \)-valued functions on \([0, \infty)\) of bounded variation and which are continuous from the right on \((0, \infty)\), respectively. \( \bar{\mu} \) denotes the total variation of the measure \( \mu : \bar{\mu}(dt) := \sum_{i=1}^q \mu_i(dt) \). \( \mathcal{L} \times \mathcal{B} \) is the product \( \sigma \)-field, where \( \mathcal{L} \) denotes the Lebesgue subsets of \([0, \infty)\) and \( \mathcal{B} \) denotes the Borel subsets of \( \mathbb{R}^q \). \( B \) is the open unit ball in Euclidean space.

The addressed class of problems arise in a wide variety of application areas such as finance, impact vibro-mechanics, management of renewable resources, and aerospace navigation, (consider \([10, 5, 17, 9, 20]\), to name just a few references), for which the solution should be found within the set of control processes with trajectories of bounded variation. Naturally, this fuelled the relatively recent rapid development of a, by now, considerable body of theory for this class of systems (see, for example, \([1, 7, 8, 13, 15, 16, 18, 21, 22, 23, 26, 27, 28, 29, 30, 31, 33]\), and references therein) and supporting control strategies computation schemes, \([14, 3, 4, 19]\).

Furthermore, the pervasive availability of computational capabilities associated to technological evolution led to the emergence of control systems paradigms, generically designated by Hybrid Systems, whose description involves a combination of (continuum) time driven and by event dynamics, \([32, 6, 12, 25]\). Albeit in a modelling context that differs from the one adopted in this article, \([2]\) draws the attention for the fact that the impulsive control framework is suitable to capture important features of hybrid systems. Therefore, besides the motivation naturally associated with the above mentioned range of applications addressed by impulsive control, its relevance in dealing with systems of hybrid nature (in the sense defined above) depends on the type of results and algorithms that can be developed in this framework. This is an additional incentive to pursue the research issues addressed in this article.

This article is organized as follows: in the next section we introduce the solution concept. Then, together with the presentation of relevant preliminary conventional definitions and results, we present, in section 3, both weak and strong invariance conditions as well as an outline of the proofs of the main results. Finally, a couple of examples are presented in section 4.
2 Solution concept

For the concept of solution we use that introduced in ([23, 24], making the necessary changes to encompass the unbounded interval \([0, 1]\)). This concept has some important robustness properties. For this we need to describe a change of variables technique. Now, we introduce the following functions that will be needed for the change of variables:

- \(M_i(t) := \begin{cases} \int_{[0,t]} \mu_i(ds), & \forall t > 0 \\ 0, & \text{if } t = 0 \end{cases} \) if \(i = 1, \ldots, q\),

- \(\eta(t) := t + \sum_{i=1}^{m} M_i(t)\),

- \(\tilde{\eta}(t) := \begin{cases} \{\eta(t)\} & \text{if } \hat{\mu}(\{t\}) = 0, \\ [\eta(t^-), \eta(t)] & \text{if } \hat{\mu}(\{t\}) > 0. \end{cases}\)

The above defined function \(\eta\) is a reparameterization of the time variable \(t\). Now we introduce the graph completion notion for the set-valued measure \(\mu\).

**Definition 2.1** A family of graph completions associated to the set-valued measure \(\mu\) is the set of the pairs \((\theta, \gamma) : [0, 1) \to K\), where \(\theta : [0, 1) \to [0, \infty)\) is the “inverse” of \(\tilde{\eta} : [0, \infty) \to \mathcal{P}([0, 1])\) in the sense that \(\theta(s) = t\), \(\forall s \in \tilde{\eta}(t)\) and \(\gamma : [0, \infty) \to \mathbb{R}^q\) is defined \(\forall s \in \tilde{\eta}(t), \forall t \in [0, \infty)\), by

\[
\gamma(s) := \begin{cases} M(\theta(s)) & \text{if } \hat{\mu}(\{t\}) = 0 \\ M(t^-) + \int_{\theta(t^-)}^{s} v(\sigma)d\sigma & \text{if } \hat{\mu}(\{t\}) > 0, \end{cases}
\]

for some \(v(\cdot) \in V^t\). Here, \(M(\cdot) := \text{col}(M_1(\cdot), \ldots, M_q(\cdot))\) and

\[V^t := \{v : \tilde{\eta}(t) \to K| \dot{\theta}(s) + \sum_{i=1}^{m} v_i(s) = 1 \forall s \in \tilde{\eta}(t), \int_{\eta(t)} v(s)ds = \mu(\{t\})\}.\]

We shall need the following change of variables lemma.

**Lemma 2.1** Let \((\theta, \gamma)\) be a family of graph completions of \(\mu \in C^*([0, 1); K)\). Then,

(i) \(\theta\) and \(\gamma\) are Lipschitz continuous, non-negative functions satisfying

\[\dot{\theta}(s) + \sum_{i=1}^{k} \dot{\gamma}_i(s) = 1 \quad \mathcal{L} - a.e..\]
(ii) For all Borel measure $\mu \in C^*(]0,\infty[, K)$, all integrable function $G : [0, \infty) \rightarrow \mathbb{R}^{n\times q}$ and Borel set $T \subset [0, \infty)$, we have

$$\int_{\theta^{-1}(T)} G(\theta(s)) \gamma(s) ds = \int_T G(\tau) \mu(d\tau).$$

(iii) For all Lebesgue measurable function $f : [0, \infty) \rightarrow \mathbb{R}^n$ and Borel set $S \subset [0, \infty)$, $\theta(S)$ is also Borel set and

$$\int_S f(\theta(s)) \delta(s) ds = \int_{\theta(S)} f(\tau) d\tau.$$

Finally, we introduce the concept of robust solution.

**Definition 2.2** The trajectory $x$, with $x(0) = x_0$, is admissible for (1) if $x(t) = x_{ac}(t) + x_a(t)$ $\forall t \in [0, \infty)$, where

$$\begin{cases}
\dot{x}_{ac}(t) \in F(t, x(t)) + G(t, x(t)) \cdot w_{ac}(t) \text{ a.e.} \\
x_a(t) = \int_{[0, t]} g_c(\tau) w_c(\tau) d\mu_{ac}(\tau) + \int_{[0, t]} g_a(\tau) d\mu_{sa}(\tau).
\end{cases}$$

Here, $\mu$ is the total variation measure associated with $\mu_{ac}$, $\mu_{sa}$ and $\mu_{ac}$ are, respectively, the singular continuous, the singular atomic, and the absolutely continuous components of $\mu$, $w_{ac}$ is the time derivative of $\mu_{ac}$, $w_{ac}$ is the Radon-Nicodym derivative of $\mu_{ac}$ with respect to its total variation, $g_c(\cdot)$ is a $\mu_{ac}$ measurable selection of $G(\cdot, x(\cdot))$ and $g_a(\cdot)$ is a $\mu_{sa}$ measurable selection of the multifunction

$$\bar{G}(t, x(t^-); \mu(\{t\})) : [0, \infty) \times \mathbb{R}^n \times K \rightarrow \mathcal{P}(\mathbb{R}^n)$$

that takes, as values, the set of all $\xi(\eta(t))$ where $(\xi(\cdot), \gamma_\mu(\cdot))$ satisfies:

$$\begin{align}
\xi(\eta(t^-)) &= x(t^-), \\
\dot{\xi}(s) &\in G(t, \xi(s)) \gamma_\mu(s) \text{ a.e. in } \eta(t), \\
\mu(\{t\}) &= \gamma_\mu(\eta(t)) - \gamma_\mu(\eta(t^-)),
\end{align}$$

for some function $G \in \mathcal{G}$ continuous in $t$ and Lipschitz in $x$. Here, $(\xi, \gamma_\mu)$ belongs to $AC([0, \infty); \mathbb{R}^n \times \mathbb{R}^n)$, and the pair $(\theta, \gamma_\mu)$ is a graph completion of $\mu$.

**Remark.** We treat the trajectories of (1) like a multi-valued arc. We consider their images like curves in $\mathbb{R}^n$ for each time $t$. We denote such a trajectory by $x_t(\cdot)$. When $t$ is a continuity point of the measure control $\mu$, $x_t(\cdot)$ is a singleton, while if $t$ is an atom of the control measure $\mu$, $x_t(\cdot)$ is regarded as a set of curves:

$$x_t(\cdot) := \{\xi : \eta(t) \rightarrow \mathbb{R}^n : \xi \text{ satisfies (2)-(4)}\}.$$

Let $\{t_i\}_{i=1}^\infty$ be a sequence of atoms of $\mu$ and $(x, \mu)$ be a feasible process. Then, $x_{t_i}(\cdot) \in \mathcal{S}$ means

$$x(t) \in S, \ \forall t \in [0, \infty) \text{ and } x_{t_i}(s) \subset S, \ \forall s \in \eta(t_i).$$
We say that \((x, \mu)\) is a feasible process for system (1) if \(\mu \in C^*(]0, \infty[; K)\) and \(x\) is robust solution of (1).

By using a change of variables technique, we can define a conventional differential inclusion associated to the impulsive differential inclusion. The theorem below establishes a connection between the solutions of both these differential inclusions. The proof is similar to that given to a similar result (see Theorem 4.1 of [29]) in which the control measure is scalar-valued and, therefore, we omit it.

**Theorem 2.1** Suppose that the multi-functions \(F\) and \(G\) satisfy:

- \(F\) takes values in closed sets and is \(L \times B\)-measurable.
- \(G\) takes values in closed sets and is Borel-measurable.

Fix a measure \(\mu \in C^*(]0, \infty[; K)\) and a initial value \(x_0\). Let \((\theta, \gamma_\mu)\) be the graph completion of \(\mu\) and \(\eta\) the reparameterization function.

(i) Suppose that \(x(\cdot) \in BV^+([0, \infty[; \mathbb{R}^n)\) is a robust solution of (1) (with respect to \(\mu\) and \(x_0\)). Then, there is a solution \(y(\cdot) \in AC([0, \infty[; \mathbb{R}^n)\) to

\[
\begin{aligned}
\dot{y}(s) &\in F(\theta(s), y(s))\dot{\theta}(s) + G(\theta(s), y(s))\dot{\gamma}_\mu(s) \\
y(0) &= x_0
\end{aligned}
\]  

(5)

for which

\[x(t) = y(\eta(t)) \quad \text{for all } t \in (0, \infty).\]  

(6)

Conversely,

(ii) Suppose that \(y(\cdot) \in AC([0, \infty[; \mathbb{R}^n)\) is a solution to (5). Then there exists a solution \(x(\cdot) \in BV^+([0, \infty[; \mathbb{R}^n)\) to (1) for which (6) is satisfied.

(iii) Take a solution \(x\) to (1). Let \(y\) be a solution to (5) such that (6) is satisfied.

Then

\[\|x\|_{T.V.} \leq \|y\|_{T.V.}\]

In the sequel, we will denote functions and variables of the extended reparameterized system by \(\tilde{\cdot}\), i.e., we have \(\tilde{x} \in \tilde{F}(x)\) where \(\tilde{x} = \text{col}(x^0, x)\) and

\[
\tilde{F} := \{\text{col}(v_0, Fv_0 + Gv) : \text{col}(v_0, v) \in \tilde{V}\}.
\]

In this context \(\tilde{S} = [0, \infty) \times S\).

## 3 Invariance Results

In this section we state the results of invariance for impulsive control systems. Let us first introduce the notion of invariance informally.

Let \(S \subset \mathbb{R}^n\) be a closed set and \(F\) be a set valued map on the \((t, x)\) space specifying the dynamics in a differential inclusion form. We say that a certain system \((\tilde{F}, S)\) is invariant when all or some of the trajectories of \(F\) that start in \(S\) remain in this set for all future times.
Definition 3.1 Let $S \subset \mathbb{R}^n$. We say that the system $((F, G), S)$ is weakly invariant if $\forall x_0 \in S$ there exists a feasible process $(x(\cdot), \mu(\cdot))$ of (1) with $x(0) = x_0$ and $x(t) \in S$ for all $t \geq 0$ and, $\forall i \in \mathbb{N}$, $\exists x_i(\cdot) \subset x_i(t)$ such that $x_i(s) \in S$ for all $s \in \bar{\eta}(t_i)$.

Definition 3.2 If $x_t(\cdot) \subset S \forall t \geq 0$, for all feasible process $(x(\cdot), \mu(\cdot))$ of (1) such that $x(0) \in S$, we say that the system $((F, G), S)$ is strongly invariant.

Definition 3.3 The attainable set $A(x_0; T)$ from $x_0$ at the time $T$ is given by:

$A(x_0; T) := \{x(T) : (x, \mu) \text{ is a feasible process of (1) with } x(0) = x_0\}$.

The results presented in this section require a set of assumptions on the multi-functions $F$ and $G$ that we call by Standing Hypotheses. These hypotheses will hold hereafter.

Standing Hypotheses:

(h1) For every $x \in \mathbb{R}^n$, $F(x)$ and $G(x)$ are convex, compact and non-empty sets.

(h2) $F$ and $G$ are upper semicontinuous.

(h3) There are constants $a$ and $b$ such that for every $x \in \mathbb{R}^n$,

\[ v \in F(x) \implies \|v\| \leq a\|x\| + b; \quad (7) \]

\[ v \in G(x) \implies \|v\| \leq a\|x\| + b; \quad (8) \]

The hypothesis (h3) is known as linear growth condition.

We recall that $F$ is upper semicontinuous at $x$ if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$\|y - x\| < \delta \implies F(y) \subset F(x) + \varepsilon B$.

Before stating some equivalent forms to the weak invariance of the system $((F, G), S)$, a result that generalizes the one for conventional control problems to the impulsive context, we will introduce the following assumption.

\[ x^0(0) = 0 \text{ and } \lim_{s \to \infty} x^0(s) = \infty. \quad (9) \]

Notice that this condition is naturally satisfied if the total variation measure $\bar{\mu}$ of the control measure $\mu$ satisfies

$\forall T > 0, \lim_{t \to \infty} \bar{\mu}([t, t + T]) = 0.$

Proposition 3.1 The system $(\bar{F}, \bar{S})$ is weakly invariant if and only if the system $((F, G), S)$ is also weakly invariant.
process \((\text{col}(x^0, y))\) of \(\tilde{F}\) such that \(y(0) = x_0 \in S\) and \(y(s) \in S\), \(\forall s \geq 0\). By Theorem 2.1, there exists a process \((x, \mu)\) of \((F, G)\) such that \(x(t) = y(\eta(t)), \forall t \geq 0\). Thus \(x(0) = x_0 \in S\) and \(x(t) \in S, \forall t \geq 0\), i.e., \(((F, G), S)\) is weakly invariant.

\[\begin{align*}
&\Rightarrow \text{ Suppose that the system } (\tilde{F}, \tilde{S}) \text{ is weakly invariant. Let } x_0 \in S. \text{ Then, } \\
&\text{there exists a trajectory } \text{col}(x^0, y) \text{ of } \tilde{F} \text{ such that } y(0) = x_0 \in S \text{ and } y(s) \in S, \\
&\forall s \geq 0. \\
&\text{By Theorem 2.1, there exists a process } (x, \mu) \text{ of } (F, G) \text{ such that } \\
&x(t) = y(\eta(t)), \forall t \geq 0. \text{ Thus } x(0) = x_0 \in S \text{ and } x(t) \in S, \forall t \geq 0, \text{ i.e., } \\
&((F, G), S) \text{ is weakly invariant.}
\end{align*}\]

\[\begin{align*}
&\Leftarrow \text{ Let } x_0 \in S. \text{ If } ((F, G), S) \text{ is weakly invariant, then there exists a feasible } \\
&\text{process } (x, \mu) \text{ of } (1) \text{ with } x(0) = x_0, x(t) \in S \forall t \geq 0, \text{ and, } \\
&\forall i, \exists \xi_i(\cdot) \in x_t(\cdot) \text{ such that } \xi_i(s) \in S, \text{ for all } s \in \tilde{\eta}(t_i). \text{ Again, by Theorem 2.1, there exists a } \\
&\text{trajectory } \text{col}(x^0, y) \text{ of } \tilde{F} \text{ satisfying } (6). \text{ Then, by putting } \\
y(s) = \begin{cases} \\
\quad x(\theta(s)), & \text{if } s \in [0, \infty) \setminus \bigcup_{i=1}^{\infty} \tilde{\eta}(t_i) \\
\quad \xi_i(s), & \text{if } s \in \bigcup_{i=1}^{\infty} \tilde{\eta}(t_i),
\end{cases}
\end{align*}\]

we have that \(y(s) \in S \forall s \geq 0\) and \(y(0) = x_0\), and, therefore, \((\tilde{F}, \tilde{S})\) is weakly invariant.\)

Before pursuing, let us introduce the definition of proximal normal cone (see [11]) used in the next result.

Let \(S \subset \mathbb{R}^n\) be a closed set and take \(x_0 \in S\). Then \(\zeta \in \mathbb{R}^n\) is a proximal normal vector to \(S\) at \(x_0\) if \(\exists \alpha > 0\) such that

\[d_S(x_0 + \alpha \zeta) = \alpha \|\zeta\|\]

where \(d_S(\cdot)\) is the distance function given by \(d_S(y) := \inf\{\|y-s\| : s \in S\}\).

The proximal normal cone to \(S\) at \(x_0\), \(N^p_S(x_0)\), is the set of all proximal normals of \(S\) at \(x_0\).

**Theorem 3.1** Suppose that the condition (9) holds. Then,

\[\begin{align*}
\begin{cases} \\
\text{For each } \tilde{x} \in \tilde{S}, \text{ there exist } (v_0, v) \in V, & f \in F(x) \text{ and } G \in G(x) \text{ s. t. } \\
\sum_{i=1}^{q} v_i g_i(\zeta_1, \zeta_2, \ldots, \zeta_q), \zeta \leq 0, & \forall \zeta \in N^p_S(\tilde{x}) \\
\text{if and only if } \\
The \text{ system } ((F, G), S) \text{ is weakly invariant.}
\end{cases}
\end{align*}\]

**Proof.**

The proof of this result follows the structure of the corresponding result for conventional systems in [11].

In this reference, it is shown that (10) is implied by

\[\tilde{F}(x) \cap \text{col}^B_{\tilde{S}}(\tilde{x}) \neq \emptyset \forall \tilde{x} \in \tilde{S}.\]

\[\text{(12)}\]

\[\begin{align*}
\text{1)} T^B_S(x) \text{ is the Bouligand tangent cone to } S \text{ at } x \text{ defined as } \\
T^B_S(x) := \left\{ \lim_{i \to \infty} \frac{x_i - x}{\lambda_i} : x_i \to x, \lambda_i \downarrow 0 \right\},
\end{align*}\]

where \(x_i \to x\) means \(x_i \in S\) and \(x_i \to x\).
which, in turn, is implied by
\[ \tilde{F}(x) \cap T^B_S(\tilde{x}) \neq \emptyset \forall \tilde{x} \in \tilde{S}. \]  

Furthermore, it is proved that (11) is equivalent to
\[ \forall x_0 \in S, \; \forall \varepsilon > 0, \; \exists \delta \in (0, \varepsilon) \; \text{s.t.} \; \mathcal{A}(x_0; \delta) \cap S \neq \emptyset. \]  

Therefore, in order to complete the proof, we only need to show that (10) implies (11) and that (14) implies (13).

Observe that \( T^B_S(\tilde{x}) = \mathbb{R} \times T^B_S(x) \) and, as a consequence (13) is equivalent to
\[ \tilde{F}(x) \cap T^B_S(x) \neq \emptyset, \; \forall x \in S, \]  

being \( \tilde{F} := \{ Fv_0 + Gv : \text{col}(v_0, v) \in \tilde{V} \} \). It can also be easily concluded that \( N^p_S(\tilde{x}) = \{0\} \times N^p_S(x) \) and, therefore, (10) is equivalent to
\[ \exists (v_0, v) \in \tilde{V}, \; f(x) \in \tilde{F}(x), \; G(x) \in \mathcal{G}(x) \; \text{s.t.} \]
\[ \langle v_0 f + \sum_{i=1}^N v_i g_i, \zeta \rangle \leq 0, \; \forall \zeta \in N^p_S(x) \forall x \in S. \]  

Now let us start with the first implication. Consider the system \((\tilde{F}(x), S)\), and let \( x_0 \in S \).

From (10), and, obviously (16), it follows that
\[ h(x, \zeta) \leq 0, \; \forall \zeta \in N^p_S(x), \]
where \( h(x, p) := \min \{ \langle w, p \rangle : w \in \tilde{F} \} \). This allows us to use Theorem 4.2.4 of [11] to find a trajectory \( y(\cdot) \) of \( \tilde{F} \) in \([0, \infty)\) such that \( y(0) = x_0 \) and \( y(s) \in S \) for all \( s \geq 0 \). Now, we only have to construct a process \((x, \mu)\) of (1) with \( x(t) = y(\eta(t)) \) for a suitable function \( \eta \) such that \( x(0) = x_0, \; x(t) \in S \forall t \geq 0, \) and \( \xi_{\mu}(s) \in S \forall s \in \eta(t_i) \), for all atoms \( t_i \) of \( \mu \) and some function \( \xi(\cdot) \in \xi_{\mu}(\cdot) \). This can be done as in Proposition 3.1 and, therefore, the system \((\tilde{F}, \mathcal{G}, S)\) is weakly invariant.

Now, we show that (14) \( \Rightarrow \) (13) or, after the remark earlier in this proof, its lower dimensional equivalent (15). First notice that (11) is equivalent to (14). So, by Proposition 3.1, we can conclude that the claim (14) holds also to the attainable set for the absolutely continuous case
\[ \hat{y}(s) \in F(y(s))\hat{\theta}(s) + G(y(s))\hat{\gamma}(s). \]

This allows us to adapt the proof of the similar result for the conventional control problem in [11]. Suppose that (14) holds for the absolutely continuous version of the attainable set, which we denote here by \( \mathcal{A}^p(x_0, t) \). Then, for all \( n \in \mathbb{N} \), \( \exists \delta_n \in (0, 1/n) \) with \( \mathcal{A}^p(x_0, \delta_n) \cap S \neq \emptyset \). Therefore, for every \( n \),
\[ \hat{y}_n(s) \in F(y_n(s))\hat{\theta}_n(s) + G(y_n(s))\hat{\gamma}_n(s) \]
\[ \subset \{ F(y_n(s))v_0 + G(y_n(s))v : (v_0, v) \in \tilde{V} \} \]
\[ = \tilde{F}(y_n(s)). \]
Here \( (\theta_n, \gamma_n) \) are graph completion of measures \( \mu_n \in C^*([0, \infty); K) \).

The functions \( y_n \) have the same Lipschitz constant \( K \), so that

\[
\frac{\|y_n(\delta_n) - x_0\|}{\delta_n} \leq K, \quad \forall n.
\]

Thus, taking a subsequence (no relabelling), there exists \( v \in \mathbb{R}^n \) such that

\[
v := \lim_{n \to \infty} \frac{y_n(\delta_n) - x_0}{\delta_n}.
\]

That is, \( v \in T_{x_0}^K \). Then we need only to show that \( v \in \bar{F}(x_0) \) in order to deduce (13). We can write

\[
y_n(\delta_n) - x_0 = \int_0^{\delta_n} \dot{y}_n(s) ds.
\]

We have that \( \bar{F} \) is upper semi-continuous. Let \( \varepsilon \geq 0 \). Then for a sufficiently large, follows

\[
y_n(\delta_n) - x_0 = \int_0^{\delta_n} \{ \bar{F}(x_0) + \varepsilon B \} ds.
\]

By dividing by \( \delta_n \) and passing to the limit when \( n \to \infty \) we obtain

\[
v \in \bar{F}(x_0) + \varepsilon B.
\]

The result is obtained by taking into account that \( \varepsilon \) is arbitrary. \( \Box \)

**Proposition 3.2** The system \((\bar{F}, \bar{S})\) is strongly invariant if and only if the system \(((F, G), S)\) is also strongly invariant.

**Proof**

[\( \Rightarrow \)] Let \( (x, \mu) \) a feasible process for (1) such that \( x(0) \in S \). Then, by Theorem 2.1, there exists a trajectory \( y \) for \( f(x)\theta + g(x)v \subset \bar{F}(x) \) such that \( y(0) = x(0) \in S \), \( y(s) = x(\theta(s)) \) for all \( s \in [0, \infty) \setminus \cup_{i=1}^{\infty} \tilde{\gamma}(t_i) \) \( \& y(s) \in x_{t_i}(s) \) for all \( i \in \mathbb{N} \).

Let \( \psi(t) = \theta(s) \) and notice that \( \tilde{y} = \text{col}(\psi, y) \) is a trajectory for \( \tilde{F} \) satisfying \( \tilde{y}(0) \in \tilde{S} \). Since \( (\bar{F}, \bar{S}) \) is strongly invariant, we have that \( y(s) \in S \) for all \( s \geq 0 \).

But \( y(s) \) is any curve in \( x_{t_i} \) that possesses the same properties. Then, \( x_{t_i}(s) \subset S \) for all \( i \in \mathbb{N} \). Therefore, it follows from \( x(t) = y(\tilde{\gamma}(t)) \) that \( x_{t_i}(s) \subset S \) for all \( t \geq 0 \), i.e., \( (F, G), S) \) is strongly invariant.

[\( \Leftarrow \)] Let \( \tilde{y} \) be an arbitrary trajectory of \( \tilde{F} \) with \( \tilde{y}(0) \in \tilde{S} \). We will show that, for any arbitrary \( T > 0 \), \( y(T) \in S \). We can construct a feasible process \( (x, \mu) \) of \((F, G)\) in \([0, T]\), with \( x(0) = y(0) \in S \) just as in the proof of Theorem 3.1.

Let \( T^* := \eta(T) \). By assumption \(((F, G), S)\) is strongly invariant, and, therefore, \( x_{t_i}(s) \subset S \) for all \( t \in [0, T] \). By construction, we have \( y(s) = \theta(s) \) for all \( s \in [0, T^*] \setminus \cup_{i=1}^{\infty} \tilde{\gamma}(t_i) \) and \( y(s) \in x_{t_i}(s) \) for all \( s \in \tilde{\gamma}(t_i) \). Then, \( y(s) \in S \) for all \( s \in [0, T^*] \). But \( T^* \geq T \) and this implies that \( y(T) \in S \). Thus \((F, S)\) is strongly invariant, since \( \tilde{y} \) is an arbitrary solution of (5). \( \Box \)

In the next result, that is a generalization of the similar result for the regular case (see e.g. [11]), we need of the Lipschitz condition for multi-functions. We
say that a multi-function $\Gamma : \mathbb{R}^n \rightharpoonup \mathbb{R}^m$ is locally Lipschitz if for each $x_0 \in \mathbb{R}^n$, there exists $\delta, K > 0$ such that

$$\Gamma(x) \subset \Gamma(y) + K||x - y||B \quad \forall x, y \in x_0 + \delta B.$$ 

**Theorem 3.2** Suppose that $F$ and $G$ are locally Lipschitz. Then

$$\forall \bar{x} \in \tilde{S}, \forall (v_0, v) \in \bar{V}, \text{ we have:}$$

$$\max_{f \in F(x), \ G \in G(x)} \langle v_0 f + \sum_{i=1}^{2} v_i g_i, \bar{\zeta} \rangle \leq 0 \quad \forall \bar{\zeta} \in \mathcal{N}_{\bar{x}}^F(\bar{x})$$

if and only if

The system $((F, G), S)$ is strongly invariant. (18)

**Remark.** Following arguments in [11] it is straightforward to show that alternative equivalent characterizations of strong invariance are:

(a) $\tilde{F}(\bar{x}) \subset T_C^S(\bar{x}) \forall \bar{x} \in \tilde{S}^2$.

(b) $\tilde{F}(\bar{x}) \subset T_B^S(\bar{x}) \forall \bar{x} \in \tilde{S}$.

(c) $\tilde{F}(\bar{x}) \subset \text{co}T_B^S(\bar{x}) \forall \bar{x} \in \tilde{S}$.

(d) $\forall x_0 \in S, \exists > 0$, such that $A(x_0, t) \subset S \forall t \in [0,\varepsilon]$.

**Proof**

Since it follows from Theorem 4.3.8 of [11] that (17) is a necessary and sufficient condition for strong invariance of the system $((\tilde{F}, S), (\tilde{F}, S))$ (as well as conditions (a) – (d) in the above remark). Then, the conclusion follows immediately from Proposition 3.2. □

4 **Examples**

4.1 **Example 1**

Let $x = (x_1, x_2) \in \mathbb{R}^2$. We will verify the invariance of the system $((F, G), S)$ with

- $F(x) = \{(|x_2| - u - \frac{1}{2}, u) : -1 \leq u \leq 1\}$;
- $G(x) = \left\{\left(-\frac{x_1}{|x_1|}, k\right) : k \geq 0\right\}$;

\footnote{The Clarke tangent cone to $S$ at $x$, denoted by $T_C^S(x)$, is given by: $T_C^S(x) := \{v \in \mathbb{R}^n : d_C^S(x; v) \leq 0\}$, where $d_C^S(x; v)$ is the generalized directional derivative of $d_C^S(\cdot)$ at $x$, in the direction $v$. This is defined by $f^0'(x; v) = \limsup_{y \to x, \ t \downarrow 0} \frac{f(y + tv) - f(y)}{t}$.

\begin{align*}
\end{align*}}
• $S = \{(x_1, x_2) : -2 \leq x_1 \leq 2\}$.

In this case we have

$$N^p_S(x_1, x_2) = \left\{ \left( \frac{x_1}{|x_1|}, 0 \right) : \lambda \geq 0 \right\},$$

for $x_1 = \pm 2$. For any other pair $(x_1, x_2)$, $N^p_S(x_1, x_2) = \{(0, 0)\}$ and the inequality (16) is clearly satisfied. In the first case, by taking $(v_0, v_1) = (0, 1) \in V$, it follows that, for all values of $u$,

$$\left\langle v_0 \left( |x_1| - u - \frac{1}{2}, u \right) + v_1 \left( -\frac{x_1}{|x_1|}, k \right), \left( \frac{x_1}{|x_1|}, 0 \right) \right\rangle = -\lambda \leq 0,$$

and, therefore, (16) is satisfied. Thus, by Theorem 3.1 the system is weakly invariant.

Now we study with more details the case with $u = k = 1$. In this case, the trajectories are given by

$$\begin{align*}
  x_1(t) &= \int |x_2(t)| dt - \frac{3}{2} t + A, \\
  x_2(t) &= t + B,
\end{align*}$$

where $A$, $B$ are arbitrary constants that can be found by initial conditions. Take

$$\mu(dt) := \delta_{t_i}(t) dt$$

as the measure control, where $\delta_{t_i}$ is the unitary impulsive function at $t = t_i$. Here $\{t_i\}_{i \in \mathbb{N}}$, the sequence of atoms of the measure control, are such that either

(1) $x_1(t_i) = -2$ and $-\frac{3}{2} \leq x_2(t_i) \leq \frac{3}{2}$

or

(ii) $x_1(t_i) = 2$.

Items (i) and (ii) above mean that the trajectory reaches the boundary of $S$ in a region where the non-singular field “point outwards $S$”. At this moment the singular field is active pushing the trajectory inside $S$ and yielding a discontinuity in the trajectory. The curves in $x_{t_i}$, that connect the discontinuity points of the trajectory, are given by

$$\begin{align*}
  \xi_1(t) &= t - 2 - \eta(t_i^-) \\
  \xi_2(t) &= t + x_2(t_i^-) - \eta(t_i^-)
\end{align*}$$

if $x_1(t_i) = -2$ or

$$\begin{align*}
  \xi_1(t) &= -t + 2 + \eta(t_i^-) \\
  \xi_2(t) &= t + x_2(t_i^-) - \eta(t_i^-)
\end{align*}$$

if $x_1(t_i) = 2$. 
if \( x(t_i) = 2 \), where \( \eta \) is the time reparameterization function defined in Section 2. Since \( \eta(t_i) - \eta(t^-_i) = 1 \) (recall that the control measure \( \mu \) is a unitary impulsive function), we have that
\[
\xi_1(\eta(t_i)) = \eta(t_i) - 2 - \eta(t^-_i) = -1
\]
if \( x_1(t_i) = -2 \) and
\[
\xi_1(\eta(t_i)) = -\eta(t_i) + 2 + \eta(t^-_i) = 1
\]
if \( x_1(t_i) = 2 \). This means that the system always drives the trajectory towards the lines \( x_1 = -1 \) or \( x_1 = 1 \) when \( x_1(t^-_i) = -2 \) or \( x_1(t^-_i) = 2 \), respectively.

Figure 1 shows the non-singular field in \( S \) and a typical trajectory starting at \((-2, -3)\).

4.2 Example 2

Now we study the invariance of the system \(((F, G), S)\) with
- \( F(x) = Ax \),
- \( G(x) = Bx \), and
- \( S = \{ x : x_1^2 + x_2^2 \leq 9 \text{ and } x_2 \leq 2 \} \),

where
\[
A = \begin{bmatrix} \alpha & -2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\beta & 0 \\ 0 & -\beta \end{bmatrix},
\]
with \( \alpha, \beta \in \mathbb{R}, \beta \neq 0 \) and \( x = \text{col}(x_1, x_2) \).

By theorem 3.1 the inequality (16) must be satisfied for all \( x \in S \). Choose \( \alpha = 2 \) and \( \beta = 1 \). For \( x = (-\sqrt{5}, 2) \) we have
\[
N^p_S(x) = \{ \lambda \zeta : \zeta_1 = -\frac{\sqrt{5}}{2} \zeta_2 \text{ and } -\zeta_1, \zeta_2, \lambda \geq 0 \},
\]
where \( \zeta = (\zeta_1, \zeta_2) \). Take \((v_0, v_1) = (0, 1) \in \tilde{V}\). Then
\[
\langle v_0 f + v_1 g, \lambda \zeta \rangle = \langle (\sqrt{5}, -2), (\lambda \zeta_1, \lambda \zeta_2) \rangle = -\frac{9}{2} \lambda \zeta_2 \leq 0.
\]
For \( x = (\sqrt{5}, 2) \) we have
\[
N^p_S(x) = \{ \lambda \zeta : \zeta_1 \leq \frac{\sqrt{5}}{2} \zeta_2 \text{ and } \zeta_1, \zeta_2, \lambda \geq 0 \},
\]
where \( \zeta = (\zeta_1, \zeta_2) \). Take \((v_0, v_1) = \left(\frac{1}{2}, \frac{1}{2}\right) \in \tilde{V}\). Then
\[
\langle v_0 f + v_1 g, \lambda \zeta \rangle = \frac{1}{2} \left( \langle (\sqrt{5} - 4, \sqrt{5} - 2), \lambda (\zeta_1, \zeta_2) \rangle \right)
\leq \frac{1 - 2\sqrt{5}}{4} \lambda \zeta_2
\leq 0.
\]
For \( x = (x_1, 2) \), with \(-\sqrt{3} < x_1 < \sqrt{3}\), we have

\[ N(x) = \{ (0, \lambda) : \lambda \geq 0 \}. \]

Take \((v_0, v_1) = (0, 1) \in \dot{V}\). Then

\[ \langle v_0 f + v_1 g, (0, \lambda) \rangle = \langle (-x_1, -2), (0, \lambda) \rangle = -2\lambda \leq 0. \]

For others \( x \in S \) we have

\[ N^x_S(x) = \{ (0, 0) \} \text{ if } x \in \text{int } S \text{; or } \]

\[ N^x_S(x) = \{ \lambda(2x_1, 2x_2) : \lambda \geq 0 \} \text{ if } x \in \text{fr } S. \]

In the first case, inequality (16) is clearly satisfied. In the second case, take \((v_0, v_1) = (0, 1) \in \dot{V}\). Then

\[ \langle v_0 f + v_1 g, 2\lambda(x_1, x_2) \rangle = \langle (-x_1, -2), 2(\lambda x_1, \lambda x_2) \rangle = -2\lambda(x_1^2 + x_2^2) \leq 0. \]

Thus we have (16) satisfied for all points in \( S \). Therefore, by Theorem 3.1 the system \(((F, G), S)\) is weakly invariant so that for all \( x_0 = (x_{0_1}, x_{0_2}) \in S \) we are able to construct a process \((x, \mu)\) such that \( x(0) = x_0, x(t) \in S, \forall t \geq 0 \) and \( \exists \xi \in x_T \) with \( \xi(s) \in S \) if \( t \) is an atom of \( \mu \).

In this case with \( \alpha = 2 \) and \( \beta = 1 \) the matrix \( A \) has complex eigenvalues with positive real part, and so we have an instable foci. Hence the trajectories converge far away the origin when \( t \to \infty \). Then we need to active the singular field, because otherwise the trajectories will go outside \( S \). To do this, take

\[ \mu(dt) := l\delta_{t_i}(t) \]

as the control measure, where \( \{t_i\}_{i \in \mathbb{N}} \) is the atoms sequence and \( l \) is the size of the jump. Each \( t_i \) is such that \( x(t_i) \in \text{fr } S \). The size of the jump must be chosen in a suitable way for each atom \( t_i \). We will do this in a way that the trajectory will jump inside the region \( \{ x : x_1^2 + x_2^2 = 1 \} \subset S \). If \( x(t_i) \) is the initial state, then, for each \( i \), the trajectories of the singular field are given by

\[ \begin{align*}
\xi_1(t) &= x_1(t_i) \exp(\eta(t_i^-) - t) \\
\xi_2(t) &= x_2(t_i) \exp(\eta(t_i^-) - t).
\end{align*} \]

We wish \( \xi_1^2(\eta(t_i)) + \xi_2^2(\eta(t_i)) = 1 \). Then (recall that \( l = \eta(t_i^-) - \eta(t_i^-) \)) we choose

\[ l = \frac{\ln(x_1(t_i)^2 + x_2(t_i)^2)}{2}. \]

Figure 2 shows a trajectory starting in \( x_0 = (0, -2) \).

We can also conclude that the system \(((F, G), S)\) is weakly invariant for others values of \( \alpha \) and \( \beta \). Actually, if \( \beta > 0 \), we can, for any value of \( \alpha \), we can construct in a similar way a process \((x, \mu)\) such that \(((F, G), S)\) is invariant.
In particular, in some cases we do not need to active the singular field. For example if \( \alpha = -\sqrt{8} \). In this case, the matrix \( A \) has a double real negative eigenvalue and, therefore, the system is clearly invariant. But, if \( \beta < 0 \), we cannot say the same for all \( \alpha \): if \( \alpha = 2 \) both the conventional and the singular systems are unstable and we cannot construct a process \((x, \mu)\) such that \((F, G, S)\) is invariant, for any selection \((v_0, v_1) \in \bar{V}\).

References


Figure 1: The non-singular field in $S$ and a trajectory: the dotted parts means the “jump trajectories”.
Figure 2: The non-singular field in $S$ and a trajectory: the dashed parts mean the “jump trajectories”.