Conjectures and Theorems in the Theory of Entire Functions

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Dedicated to Professor R. S. Varga on the occasion of his 70th birthday.

Motivated by the recent solution of Karlin's conjecture, properties of functions in the Laguerre–Pólya class are investigated. The main result of this paper establishes new moment inequalities for a class of entire functions represented by Fourier transforms. The paper concludes with several conjectures and open problems involving the Laguerre–Pólya class and the Riemann ξ-function.

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1. The Laguerre-Pólya class, Karlin's conjecture and the Turán inequalities

A real entire function $\psi(x)$ is said to be in the *Laguerre-Pólya class*, written $\psi \in \mathcal{L}-\mathcal{P}$, if $\psi(x)$ can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2+\beta x} \prod_{k=1}^{\omega}(1+x/x_k)e^{-x/x_k}, \qquad (0 \le \omega \le \infty), \tag{1.1}$$

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where c, β, x_k are real, $\alpha \geq 0, m$ is a nonnegative integer, $\sum x_k^{-2} < \infty$ and where, by the usual convention, the canonical product reduces to 1 when $\omega = 0$. Pólya and Schur [24] termed a real entire function $\varphi(x)$ as a function of *type I* in the Laguerre-Pólya class, written $\varphi \in \mathcal{L}-\mathcal{P}I$, if $\varphi(x)$ or $\varphi(-x)$ can be represented in the form

$$\varphi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\omega} (1 + x/x_k), \qquad (0 \leq \omega \leq \infty), \tag{1.2}$$

where c is real, $\sigma \geq 0, m$ is a nonnegative integer, $x_k > 0$, and $\sum 1/x_k < \infty$. It is clear that $\mathcal{L}-\mathcal{P}I \subset \mathcal{L}-\mathcal{P}$. The significance of the Laguerre-Pólya class in the theory of entire functions is natural, since functions in this class, *and only these* are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros (see, for example, Levin [18, Chapter 8]). Thus, it follows from this result that the class $\mathcal{L}-\mathcal{P}I$ is closed under differentiation; that is, if $\varphi \in \mathcal{L}-\mathcal{P}I$, then $\varphi^{(n)} \in \mathcal{L}-\mathcal{P}I$ for $n \geq 0$. Another fact cogent to our presentation is the following important observation of Pólya and Schur [24]. If a function

$$\varphi(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{1.3}$$

is in $\mathcal{L}-\mathcal{P}$ and its Maclaurin coefficients $\gamma_k, \quad k = 0, 1, \ldots,$ are nonnegative, then $\varphi \in \mathcal{L}-\mathcal{P}I$. In the literature, the sequence $\{\gamma_k\}_0^\infty$ of Maclaurin coefficients of a function in $\mathcal{L}-\mathcal{P}I$ is called a *multiplier sequence* (cf. Pólya and Schur [24]). For the various properties of functions in Laguerre-Pólya class we refer the reader to [3], [4], [18, Chapter VIII], [20, Kapitel II], [21] and the references contained therein. Finally, in the sequel we will adopt the following notation.

Notation. For any $2m-2$ times continuously differentiable function f, the $m \times m$ Hankel determinant $H_m(f;x)$ is defined by

$$H_m(f;x) := \det(f^{(i+j)}(x))_{i,j=0}^{m-1} = \begin{vmatrix} f(x) & f'(x) & \cdots & f^{(m-1)}(x) \\ f'(x) & f''(x) & \cdots & f^{(m)}(x) \\ \vdots & \vdots & \vdots & \vdots \\ f^{(m-1)}(x) & f^{(m)}(x) & \cdots & f^{(2m-2)}(x) \end{vmatrix}.$$

With the foregoing terminology, Karlin's conjecture ([17, p. 390] see also [3, p. 258] regarding a misprint in [17, p. 390]) is as follows.

Karlin's Conjecture. (Karlin [17, p. 390]) Let $\varphi(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \in \mathcal{L}-\mathcal{P}$ and suppose that the Maclaurin coefficients $\gamma_k \geq 0$ for $k = 0, 1, 2, \ldots$. Then for any $q = 0, 1, 2, \ldots$ and $m = 2, 3, 4, \ldots,$

$$(-1)^{m(m-1)/2} H_m(\varphi^{(q)}; x) \geq 0 \qquad \text{for all} \;\; x \geq 0. \tag{1.4}$$

For the motivation and interesting history of Karlin's conjecture we refer the reader to [15], [14] and [5]. In these papers, the authors have independently obtained counterexamples to Karlin's conjecture and have answered an old, related question of Pólya (c. 1934). Moreover, in [15] and [5] the authors have demonstrated that Karlin's conjecture *is valid* for several subclasses of functions in $\mathcal{L}-\mathcal{P}$! As a concrete illustration, we cite here the following result of Dimitrov [15, Theorem 5] (see also, [5, Theorem 4.5]).

Theorem 1.1. ([15, Theorem 5]) Suppose $\sum_{k=0}^{\infty} \gamma_k x^k \in \mathcal{L}-\mathcal{P}I$, where $\gamma_k \geq 0$ for $k = 0, 1, 2, \ldots$. Then the entire function $\varphi(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$ is also in $\mathcal{L}-\mathcal{P}I$ and for any $q = 0, 1, 2, \ldots$ and $m = 2, 3, 4, \ldots$,

$$(-1)^{m(m-1)/2} H_m(\varphi^{(q)}; x) \geq 0 \qquad \text{for all} \quad x \geq 0. \tag{1.5}$$

A noteworthy special case of inequalities (1.5) arises when $x = 0$:

$$(-1)^{m(m-1)/2} \begin{vmatrix} \gamma_q & \gamma_{q+1} & \cdots & \gamma_{q+m-1} \\ \gamma_{q+1} & \gamma_{q+2} & \cdots & \gamma_{q+m} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{q+m-1} & \gamma_{q+m} & \cdots & \gamma_{q+2m-2} \end{vmatrix} \geq 0. \tag{1.6}$$

These inequalities can be readily deduced from the following beautiful characterization of functions in $\mathcal{L}-\mathcal{P}I$. Let $\varphi(x) := \sum_{k=0}^{\infty} \gamma_k x^k$, $\gamma_k \geq 0$, be an entire function. Then $\varphi \in \mathcal{L}-\mathcal{P}I$ if and only if $\{\gamma_k\}_0^{\infty}$ is a totally positive sequence (see [17, p. 412, Theorem 5.3] or [15]). We recall that $\{\gamma_k\}_0^{\infty}$ is a totally positive sequence if $\sum \gamma_k x^k$ is an entire function and if all the minors of all orders of the infinite lower triangular matrix $(\gamma_{i-j})_{i,j=1}$, where $\gamma_m := 0$ if $m < 0$, are nonnegative.

It has been known for a long time that Karlin's conjecture is true in certain special cases. Indeed, if $m = 2$, inequalities (1.4) reduce to the well-known *Laguerre inequalities* (see, for example, [3] or [8])

$$L_q(\varphi(x)) := (\varphi^{(q)}(x))^2 - \varphi^{(q-1)}(x)\varphi^{(q+1)}(x) \geq 0, \qquad q = 1, 2, 3 \ldots, \tag{1.7}$$

which hold for *all* real x and for *all* functions $\varphi \in \mathcal{L}-\mathcal{P}$. Thus, if $\varphi(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$ is an entire function of order at most 2, then (substituting $x = 0$ in inequality (1.7)) a *necessary condition* that $\varphi(x)$ have only real zeros is that

$$T_k := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \qquad k = 1, 2, 3, \ldots. \tag{1.8}$$

While the inequalities (1.8) are today commonly referred to as the *Turán inequalities* (associated with the entire function $\varphi(x)$), they may be more precisely called the *Euler-Laguerre-Pólya-Schur-Turán inequalities.* Karlin's conjecture is also valid in the case when $m = 3$. In [3] Craven and Csordas investigated certain polynomial invariants and used them to prove the following theorem.

Theorem 1.2. ([3, Theorem 2.13]) If

$$\varphi(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \in \mathcal{L} - \mathcal{P}I, \quad \text{where} \quad \gamma_k \geq 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, \tag{1.9}$$

then

$$H_3(\varphi^{(q)}; x) \leq 0 \qquad \text{for all} \ \ x \geq 0 \ \ \text{and} \ \ q = 0, 1, 2, \ldots. \tag{1.10}$$

In particular,

$$\Delta_{k-2} := \begin{vmatrix} \gamma_{k-2} & \gamma_{k-1} & \gamma_k \\ \gamma_{k-1} & \gamma_k & \gamma_{k+1} \\ \gamma_k & \gamma_{k+1} & \gamma_{k+2} \end{vmatrix} \leq 0, \qquad k = 2, 3, 4, \ldots. \tag{1.11}$$

An examination of the inequalities (1.11) suggests the following extension of the Turán inequalities (1.8). We will say that a sequence of nonnegative real numbers $\{\gamma_k\}_{k=0}^{\infty}$ satisfies the *double Turán inequalities*, if

$$E_k := T_k^2 - T_{k-1}T_{k+1} \geq 0, \qquad k = 2, 3, 4, \ldots. \tag{1.12}$$

Now a calculation shows that (cf. (1.11)) $E_k = -\gamma_k \Delta_{k-2}$, $k = 2, 3, \ldots$, and thus as an immediate consequence of Theorem 1.2 we have the following corollary.

Corollary 1.3. ([3, Corollary 2.14]) If

$$\varphi(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \in \mathcal{L} - \mathcal{P}I, \quad \text{where} \quad \gamma_k \geq 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, \tag{1.13}$$

then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfies the double Turán inequalities

$$E_k = T_k^2 - T_{k-1}T_{k+1} = -\gamma_k \Delta_{k-2} \geq 0, \qquad k = 2, 3, 4, \ldots, \tag{1.14}$$

where Δ_{k-2} is defined by (1.11).

For other types of generalizations of the Turán inequalities we refer to [16] or [8].

2. The Turán and double Turán inequalities for Fourier transforms

In this section we establish a new sufficient condition that guarantees that the double Turán inequalities (see (1.14) of Corollary 1.3) hold for a class of entire functions represented by Fourier transforms. While the moment inequalities derived here are of independent interest, our investigation is motivated by the theory of $\mathcal{L} - \mathcal{P}$ functions and its intimate connection with the Riemann ξ-function. For the reader's convenience and for the sake clarity of exposition, we begin with a brief review of some terminology and facts that will be needed in the sequel.

Let

$$H(x) := \frac{1}{8}\xi\left(\frac{x}{2}\right) := \int_0^\infty \Phi(t)\cos(xt)dt\,, \tag{2.1}$$

where

$$\Phi(t) := \sum_{n=1}^{\infty} \pi n^2\left(2\pi n^2 e^{4t} - 3\right)\exp\left(5t - \pi n^2 e^{4t}\right). \tag{2.2}$$

Then it is well known that Riemann Hypothesis is equivalent to the statement that all the zeros of $H(x)$ are real (cf. [26, p. 255]). We remark parenthetically that, today, there are no known *explicit* necessary and sufficient conditions which a function must satisfy in order that its Fourier transform have only real zeros (see, however, [23, p. 17] or [22, p. 292]). Nevertheless, the *raison d'être* for investigating the kernel $\Phi(t)$ is that there is a connection (the precise meaning of which is unknown) between the properties of $\Phi(t)$ and the distribution of the zeros of its Fourier transform.

Some of the known properties of $\Phi(t)$ defined by (2.2) are summarized in the following theorem.

Theorem 2.1. ([9, Theorem A]) Consider the function $\Phi(t)$ of (2.2) and set

$$\Phi(t) = \sum_{n=1}^{\infty} a_n(t), \tag{2.3}$$

where

$$a_n(t) := \pi n^2\left(2\pi n^2 e^{4t} - 3\right)\exp\left(5t - \pi n^2 e^{4t}\right), \qquad n = 1, 2, \ldots. \tag{2.4}$$

Then, the following are valid:

(i) for each $n \geq 1$, $a_n(t) > 0$ for all $t \geq 0$, so that $\Phi(t) > 0$ for all $t \geq 0$;

(ii) $\Phi(z)$ is analytic in the strip $-\pi/8 < Im\ z < \pi/8$;

(iii) $\Phi(t)$ is an even function, so that $\Phi^{(2m+1)}(0) = 0, \quad m = 0, 1, \ldots$;

(iv) $\Phi'(t) < 0$ for all $t > 0$;

(v) for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \Phi^{(n)}(t)\exp\left[(\pi - \varepsilon)e^{4t}\right] = 0$$

for each $n = 0, 1, \ldots$.

Thus,

$$H(x) = \frac{1}{2}\int_{-\infty}^{\infty} \Phi(t)e^{ixt}dt = \int_0^\infty \Phi(t)\cos(xt)dt \tag{2.5}$$

is an entire function of order one ([26, p. 16]) of maximal type (cf. [10, Appendix A]) whose Taylor series about the origin can be written in the form

$$H(z) = \sum_{m=0}^{\infty} \frac{(-1)^m b_m}{(2m)!} z^{2m}, \tag{2.6}$$

where

$$b_m := \int_0^{\infty} t^{2m} \Phi(t) dt, \qquad m = 0, 1, 2, \ldots. \tag{2.7}$$

The change of variable, $z^2 = -x$ in (2.6), gives

$$F(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} = \sum_{k=0}^{\infty} b_k \frac{x^k}{(2k)!}, \qquad \gamma_k := \frac{k!}{(2k)!} b_k. \tag{2.8}$$

Then it is easy to see that $F(x)$ is an entire function of order $1/2$ and that the Riemann Hypothesis is equivalent to the statement that all the zeros of $F(x)$ are real and negative, that is, $F(x) \in \mathcal{L} - \mathcal{P}I$. Thus, a necessary condition for $F(x)$ to have only real zeros is that the γ_k's satisfy the Turán inequalities. Whence, in terms of the moments b_m (cf. (2.7)), the Turán inequalities can be expressed in the form

$$b_m^2 - \frac{2m-1}{2m+1} b_{m-1} b_{m+1} \geq 0, \qquad m = 1, 2, 3, \ldots. \tag{2.9}$$

Different proofs of these inequalities may be found in [9] and [7].

Preliminaries aside, we next turn to a class of kernels whose properties parallel those of $\Phi(t)$ listed in Theorem 2.1.

Definition 2.2. A function $K : I\!R \longrightarrow I\!R$ is called an *admissible kernel*, if it satisfies the following properties:

(i) $K(t) > 0$ for $t \in I\!R$,

(ii) $K(t)$ is analytic in the strip $|Im\ z| < \tau$ for some $\tau > 0$,

(iii) $K(t) = K(-t)$ for $t \in I\!R$,

(iv) $K'(t) < 0$ for $t > 0$, and

(v) for some $\varepsilon > 0$ and $n = 0, 1, 2, \ldots$,

$$K^{(n)}(t) = O\left(\exp(-|t|^{2+\varepsilon})\right) \quad \text{as} \quad t \longrightarrow \infty.$$

Now it can be readily verified that the Fourier transform of an admissible kernel, $K(t)$, represents an even entire function whose moments

$$b_k := \int_0^{\infty} t^{2k} K(t) dt, \qquad k = 0, 1, 2, \ldots \tag{2.10}$$

all exist. The proof of the main theorem of this section (see Theorem 2.4 below) requires a preparatory result. This result provides a remarkable relationship that was discovered by the authors, thanks to a bit of serendipity, in the course of their investigation of logarithmically concave functions.

Lemma 2.3. Let $K(t)$ be an admissible kernel. For $t > 0$, set $s(t) := K(\sqrt{t})$, $f(t) := s'(t)^2 - s(t)s''(t)$ and

$$w(t) := \begin{vmatrix} \left(\frac{K'(t)}{tK(t)}\right)' & \left(\frac{K'(t)}{tK(t)}\right)'' \\ \left(\frac{(K'(t)/t)'}{tK(t)}\right)' & \left(\frac{(K'(t)/t)'}{tK(t)}\right)'' \end{vmatrix}.$$

If $f(t) > 0$ and $g(t) := (\log(f(t)))'' < 0$ for $t > 0$, then

$$w(t) = \frac{64\, t^3\, g(t^2)\, f(t^2)^2}{K(t)^4} < 0, \qquad t > 0. \tag{2.11}$$

Proof. Using the properties of an admissible kernel, the verification of (2.11) involves only routine, straightforward calculations and thus we omit the details here. However, the calculations are rather lengthy and consequently the reader may wish to use a symbol-manipulating program to check the validity of the expression for $w(t)$ given in (2.11). $\square$

Theorem 2.4. Let $K(t)$ be an admissible kernel and let b_k denote its moments defined by (cf. (1.10)).

(a) If $\log K(\sqrt{t})$ is concave for $t > 0$, that is,

$$(\log K(\sqrt{t})'' < 0 \qquad \text{for} \quad t > 0, \tag{2.12}$$

then the Turán inequalities (2.9) hold.

(b) Let $s(t) := K(\sqrt{t})$ and $f(t) := s'(t)^2 - s(t)s''(t)$. If both $\log K(\sqrt{t})$ and $\log f(t)$ are concave for $t > 0$, that is, if the inequalities (2.12) and

$$(\log f(t)))'' < 0 \qquad \text{for} \quad t > 0, \tag{2.13}$$

hold, then the double Turán inequalities (1.14) also hold, where $\gamma_k := (k!/(2k)!)b_k$.

Proof. Part (a) of the theorem was proved in [7, Proposition 2.2]. Here we first provide a slightly different proof of (a) and then, by generalizing the technique, we proceed to prove part (b).

Since $T_k = \gamma_k^2 - \gamma_{k-1}\gamma_{k+1}$ (cf. (1.8)), it is clear that

$$-T_k = c_k \begin{vmatrix} (2k-1)b_{k-1} & b_k \\ (2k+1)b_k & b_{k+1} \end{vmatrix},$$

where $c_k := 2\,k!(k+1)!/((2k)!(2k+2)!)$. If we integrate by parts (see (2.10)) and use property (v) (cf. Definition 2.2) of the admissible kernel, $K(t)$, then we obtain

$$(2k-1)b_{k-1} = \int_0^\infty t^{2k}(-K'(t)/t)dt.$$

Hence, by a problem of Pólya and Szegő [25, Part II, Problem 68], we can express $-T_k$ as

$$-T_k = \frac{c_k}{2}\int_0^\infty\int_0^\infty x_1^{2k}x_2^{2k}(x_1+x_2)K(x_1)K(x_2)\left\{(x_2-x_1)\begin{vmatrix} \frac{-K'(x_1)}{x_1K(x_1)} & \frac{-K'(x_2)}{x_2K(x_2)} \\ 1 & 1 \end{vmatrix}\right\}dx_1dx_2.$$

Now, by the mean value theorem, the expression in braces is equal to

$$(x_2-x_1)^2\left(\frac{d}{dt}\left\{\frac{K'(t)}{tK(t)}\right\}\right)_{t=\eta}, \quad \text{where} \quad \eta \in (\min\{x_1,x_2\},\max\{x_1,x_2\}).$$

Since an easy calculation shows that the assumption (2.12) is equivalent to

$$\frac{d}{dt}\left\{\frac{K'(t)}{tK(t)}\right\} < 0 \qquad \text{for} \quad t > 0.$$

and since the kernel $K(t) > 0,\ \ t \in \mathbb{R}$, we conclude that the integrand, in the integral representation of $-T_k$, is negative. Thus, the proof of part (a) is complete.

Turning to the proof of part (b), we first note that, (i) $f(t) > 0,\ \ t > 0$, by (2.12) and (ii) in view of the relations (1.14), it suffices to prove inequalities (1.11). Since $\gamma_k = \frac{k!}{(2k)!}b_k$, it follows from an elementary, albeit tedious, calculation that, for $k \geq 2$,

$$\Delta_{k-2} = c_1(k)\begin{vmatrix} (2k-1)(2k-3)b_{k-2} & (2k-1)b_{k-1} & b_k \\ (2k+1)(2k-1)b_{k-1} & (2k+1)b_k & b_{k+1} \\ (2k+3)(2k+1)b_k & (2k+3)b_{k+1} & b_{k+2}, \end{vmatrix},$$

where

$$c_1(k) := \frac{1}{(2k+1)^2(2k+3)}\left(\frac{k!}{(2k)!}\right)^3.$$

In order to express Δ_{k-2} in terms of integrals, we integrate (1.10) by parts twice and obtain

$$(2k+3)(2k+1)b_k = \int_0^\infty t^{2k+3}\left(\frac{K'(t)}{t}\right)' dt.$$

Thus,

$$\Delta_{k-2} = c_1(k)\begin{vmatrix} \int_0^\infty t^{2k-1}\left(\frac{K'(t)}{t}\right)' dt & \int_0^\infty t^{2k-1}(-K'(t))dt & \int_0^\infty t^{2k-1}(tK(t))dt \\ \int_0^\infty t^{2k+1}\left(\frac{K'(t)}{t}\right)' dt & \int_0^\infty t^{2k+1}(-K'(t))dt & \int_0^\infty t^{2k+1}(tK(t))dt \\ \int_0^\infty t^{2k+3}\left(\frac{K'(t)}{t}\right)' dt & \int_0^\infty t^{2k+3}(-K'(t))dt & \int_0^\infty t^{2k+3}(tK(t))dt \end{vmatrix}.$$

Now, another application of [25, Part II, Problem 68] (with Pólya and Szegő's notation, $f_1(t) := t^{2k-1}$, $f_2(t) := t^{2k+1}$, $f_3(t) := t^{2k+3}$, $\varphi_1(t) := (K'(t)/t)'$, $\varphi_2(t) := -K'(t)$ and $\varphi_3(t) := tK(t)$) yields the triple integral representation

$$\Delta_{k-2} = \frac{c_1(k)}{3!} \int_0^\infty \int_0^\infty \int_0^\infty x_1^{2k-1} x_2^{2k-1} x_3^{2k-1} M(x_1^2, x_2^2, x_3^2) V(K; x_1, x_2, x_3) dx_1 dx_2 dx_3,$$

where

$$M(x_1^2, x_2^2, x_3^2) := \begin{vmatrix} 1 & 1 & 1 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^4 & x_2^4 & x_3^4 \end{vmatrix} = (x_3^2 - x_2^2)(x_3^2 - x_1^2)(x_2^2 - x_1^2)$$

is the Vandermonde determinant of x_1^2, x_2^2 and x_3^2 and

$$V(K; x_1, x_2, x_3) := \begin{vmatrix} \left(\frac{K'(x_1)}{x_1}\right)' & \left(\frac{K'(x_2)}{x_2}\right)' & \left(\frac{K'(x_3)}{x_3}\right)' \\ -K'(x_1) & -K'(x_2) & -K'(x_3) \\ x_1 K(x_1) & x_2 K(x_2) & x_3 K(x_3) \end{vmatrix}.$$

Consider the above triple integral over the first octant and represent it as a sum over the regions $R_{ijs} := \{0 < x_i < x_j < x_s < \infty\}$, where the summation extends over all permutations of the indices i, j and s. For a fixed, but arbitrary permutation (i, j, s), we consider the triple integral I_{ijs} over the region R_{ijs},

$$I_{ijs} := \iiint_{R_{ijs}} x_1^{2k-1} x_2^{2k-1} x_3^{2k-1} M(x_1^2, x_2^2, x_3^2) V(K; x_1, x_2, x_3) dx_1 dx_2 dx_3.$$

The permutation (i, j, s) applied to the columns of *both* determinants $M(x_1^2, x_2^2, x_3^2)$ and $V(K; x_1, x_2, x_3)$ yields

$$I_{ijs} = \iiint_{R_{ijs}} x_i^{2k-1} x_j^{2k-1} x_s^{2k-1} M(x_i^2, x_j^2, x_s^2) V(K; x_i, x_j, x_s) dx_i dx_j dx_s. \tag{2.14}$$

We will now proceed to show that I_{ijs} is negative. To this end, with the aid of elementary operations (multiply $V(K; x_i, x_j, x_s)$ by minus one, interchange the first and the third rows of $V(K; x_i, x_j, x_s)$, and then divide each column by its first element and finally multiply the last two rows by -1), we express $V(K; x_i, x_j, x_s)$ in the form

$$V(K; x_i, x_j, x_s) = x_i x_j x_s K(x_i) K(x_j) K(x_s) W(K; x_i, x_j, x_s),$$

where

$$W(K;x_i,x_j,x_s) := \begin{vmatrix} 1 & 1 & 1 \\ -\frac{K'(x_i)}{x_iK(x_i)} & -\frac{K'(x_j)}{x_jK(x_j)} & -\frac{K'(x_s)}{x_sK(x_s)} \\ -\frac{\left(K'(x_i)/x_i\right)'}{x_iK(x_i)} & -\frac{\left(K'(x_j)/x_j\right)'}{x_jK(x_j)} & -\frac{\left(K'(x_s)/x_s\right)'}{x_sK(x_s)} \end{vmatrix}.$$

Next, we set

$$h_1(t) := 1 \qquad \text{and} \qquad h_2(t) := \frac{-K'(t)}{tK(t)}.$$

Then

$$\begin{vmatrix} h_1(t)\, h_1'(t) \\ h_2(t)\, h_2'(t) \end{vmatrix} > 0, \qquad \text{for} \quad t > 0,$$

is equivalent to (2.12). Hence, we can apply a generalized mean-value theorem [25, Part V, Problem 99] to conclude that there exists η, $x_i < \eta < x_s$, such that

$$\operatorname{sign} W(K;x_i,x_j,x_s) = \operatorname{sign} \begin{vmatrix} \left(\frac{K'(\eta)}{\eta K(\eta)}\right)' & \left(\frac{K'(\eta)}{\eta K(\eta)}\right)'' \\ \left(\frac{\left(K'(\eta)/\eta\right)'}{\eta K(\eta)}\right)' & \left(\frac{\left(K'(\eta)/\eta\right)'}{\eta K(\eta)}\right)'' \end{vmatrix} =: \operatorname{sign} w(\eta).$$

Since by Lemma 2.3 (cf. (2.11)) $w(t) < 0$ for $t > 0$, $W(K;x_i,x_j,x_s) < 0$ and *a fortiori* $V(K;x_i,x_j,x_s) < 0$ in R_{ijs}. Therefore, the integral I_{ijs} (cf. (2.14)) is negative. This completes the proof of the theorem. □

Remark 2.5. We remark that an examination of the proof of Theorem 2.4 shows that it remains valid under less restrictive assumptions on the kernel $K(t)$.

3. Open problems and conjectures

We conclude this paper with a brief survey of some open problems and conjectures pertaining to functions in the Laguerre-Pólya class ($\mathcal{L}-\mathcal{P}$) and the Riemann ξ-function.

Problem 3.1. (Higher order Turán inequalities.) Let $\{\gamma_k\}_0^\infty$, $\gamma_k \geq 0$, $k \geq 0$, be a multiplier sequence, so that $\varphi(x) := \sum_{k=0}^\infty \gamma_k \frac{x^k}{k!} \in \mathcal{L}-\mathcal{P}I$. Set

$$T_1(k) := T_1(k;\varphi) := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1}, \qquad k \geq 1$$

and

$$T_n(k) := T_n(k;\varphi) := T_{n-1}(k;\varphi)^2 - T_{n-1;}(k-1;\varphi)T_{n-1}(k+1;\varphi), \qquad k \geq n \geq 2.$$

Then, is it true that

$$T_n(k) = T_n(k; \varphi) \geq 0 \qquad \text{for} \qquad k \geq n \geq 2? \tag{3.1}$$

An affirmative answer to (3.1) would provide a set of *strong necessary* conditions for an entire function to have only real negative zeros. We have seen (cf. (1.14) of Corollary 1.3)) that the double Turán inequalities hold (i.e. the inequalities (3.1) are true when $n = 2$). Recently, Craven and the first author have shown, in a manuscript under preparation, that (3.1) has an affirmative answer for certain subclasses of multiplier sequences.

Problem 3.2. (The double Turán inequalities for the Riemann ξ-function.) Let

$$H(x) := \frac{1}{8}\xi\left(\frac{x}{2}\right) := \int_0^\infty \Phi(t)\cos(xt)dt = \sum_{m=0}^{\infty} \frac{(-1)^m b_m}{(2m)!} x^{2m}, \tag{3.2}$$

where the kernel, $\Phi(t)$, and the moments, b_m, are defined by (2.2) and (2.7) respectively. Let $\gamma_k := \frac{k!}{(2k)!} b_k$. Then we conjecture that the sequence $\{\gamma_k\}_0^\infty$ satisfies the double Turán inequalities

$$T_2(k) = T_2(k; H) \geq 0, \qquad k \geq 2. \tag{3.3}$$

In light of Corollary 1.3, the failure of inequalities (3.3) would imply that the Riemann Hypothesis is false. There are, however, more compelling reasons for the validity of this conjecture. First, in 1983, Varga *et al.* [9] have computed the first 120 moments with high degree of precision. (Since then these computations have been significantly extended by Varga *et al.* in the Department of Mathematical Sciences at Kent State University.) Numerical experiments, using these computed values of the moments, show that $T_2(k) > 0$, at least for $2 \leq k \leq 500$. Second, since the sequence $\{\frac{k!}{(2k)!}\}$ can be shown to satisfy (3.3) and since the moments, b_k, do not grow "too fast", we expect that (3.3) is true.

A different strategy for proving (3.3) could make use of Theorem 2.4. This approach requires, however, a careful examination of the concavity properties of the kernel $\Phi(t)$ in (3.2) as the next problem suggests.

Problem 3.3. (A concavity condition for $\Phi(t)$.) Let $s(t) := \Phi(\sqrt{t})$ and $f(t) = s'(t)^2 - s(t)s''(t)$. Then we conjecture that

$$(\log(f(t)))'' < 0 \qquad \text{for} \quad t > 0. \tag{3.4}$$

We hasten to remark that in [7, Theorem 2.1] it was shown that $\log \Phi(\sqrt{t})$ is concave for $t > 0$, so that $f(t) > 0$ for $t > 0$. Our initial investigation suggests that a proof of (3.4) may also require the fact that the function $\Phi(\sqrt{t})$ is convex for $t > 0$ [6, Theorem 2.12].

Problem 3.4. (Theorem 2.4 and the Laguerre-Pólya class.) State and prove an analogue of Theorem 2.4 for an arbitrary function in the Laguerre-Pólya class.

Since, in general, an arbitrary function $\varphi(x) \in \mathcal{L} - \mathcal{P}$ is not the Fourier transform of a "nice" function, a solution of Problem 3.4 might involve a different kind of integral representation of functions in $\mathcal{L} - \mathcal{P}$. One such representation, due to de Bruijn [1, Theorem 2] (which deserves to be better known) may be stated as follows. If $\varphi(x)$ is *any* function in $\mathcal{L} - \mathcal{P}$, then there is a unique, C^∞ function, $K(t)$, such that

$$e^{-x^2/2}\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t^2/2}K(t)e^{ixt}dt. \tag{3.5}$$

Thus, the *de Bruijn representation* (3.5), together with the observation (see Remark 2.5) that Theorem 2.4 remains valid if we relax some of the assumptions on the kernel $K(t)$, may well make this problem tractable.

We have seen in Section 1 that, while Karlin's conjecture is not true in general (the characterization of the class of functions in $\mathcal{L} - \mathcal{P}I$ for which it is valid remains open), it is true, however, in the special cases $m = 2$ (cf. (1.7)) and $m = 3$ (cf. (1.10)).

Problem 3.5. (The Laguerre inequalities and the Riemann-ξ function.) Let

$$F(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} = \sum_{k=0}^{\infty} b_k \frac{x^k}{(2k)!}, \qquad \gamma_k := \frac{k!}{(2k)!}b_k, \tag{3.6}$$

where the moments b_k are defined by (2.7). Then, prove that

$$L_q(F(x)) := (F^{(q)}(x))^2 - F^{(q-1)}(x)F^{(q+1)}(x) \geq 0, \quad q = 1, 2, 3\ldots, \text{ for } \underline{\text{all}}\ x \in I\!R. \tag{3.7}$$

In [13, Corollary 3.4] it was shown, as a consequence of the concavity properties of $\Phi(t)$, that (3.7) holds for all $x \geq 0$.

In order to formulate the next problem, we require some additional notation and background information. Let

$$H_t(x) = \int_0^{\infty} e^{tu^2}\Phi(u)\cos(xu)du, \qquad t \in I\!R, \tag{3.8}$$

so that $H_0(x) = H(x)$ and the Riemann ξ-function are related by (2.1). In 1950, de Bruijn [2] established that H_t has only real zeros for $t \geq 1/2$ and that if H_t has only real zeros for some real t, then $H_{t'}$ has only real zeros for any $t' \geq t$. In 1976, C. M. Newman [19] showed further that there is a real constant Λ, which satisfies $-\infty < \Lambda \leq 1/2$, such that

$$H_t \text{ has only real zeros if and only if } t \geq \Lambda. \tag{3.9}$$

In the literature, this constant Λ is now called the *de Bruijn-Newman constant.* This brings us to the following open problem conjectured by C. M. Newman [19].

Problem 3.6. (The de Bruijn-Newman constant.) Is it true that

$$\Lambda \geq 0? \tag{3.10}$$

The de Bruijn-Newman constant Λ has been investigated extensively because the truth of the Riemann Hypothesis is equivalent to the assertion that $\Lambda \leq 0$. The research activity in finding lower bounds for Λ, have been summarized in [11]. In particular, in [11] it was shown, with the aid of a spectacularly close pair of consecutive zeros of the Riemann zeta function, that $-5.895 \cdot 10^{-9} < \Lambda$.

The introduction of the function $H_t(x)$ (see (3.8)) suggests several other interesting questions, which, for the sake of brevity we will only mention in passing. Thus, one problem is the determination of the supremum of the values of t such that the Laguerre inequalities (3.7) fail for $H_t(x)$. (Note that in [12] it was shown that when $t := -0.0991$, then the Laguerre inequalities (3.7) fail for $x := -830.512\ldots$.) Another problem, which parallels Problem 3.2 and whose solution may shed light on the determination of the de Bruijn-Newman constant Λ, is as follows. For $k \geq 0$ set

$$\gamma_k(t) := \frac{k! b_k(t)}{(2k)!} \qquad \text{where} \qquad b_k(t) := \int_0^\infty e^{tu^2} u^{2k} \Phi(u) du, \qquad t \in I\!R. \tag{3.11}$$

Then determine the values of t, if any, for which the sequence $\{\gamma_k(t)\}_0^\infty$ fails to satisfy the double Turán inequalities (cf. Problem 3.2).

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