TRAPEZOIDAL CUBATURE FORMULAE AND POISSON'S EQUATION

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Abstract. The idea of extending univariate quadrature formulae to cubature formulae that hold for spaces of polyharmonic functions is employed to obtain in a new way bivariate trapezoidal cubature rules. The notion of univariate monospline is extended to functions of two variables in terms of a solution of Poisson's equation. This approach allows us to characterize the error of the trapezoidal cubature formulae. A Hermitian type cubature is also investigated.

1. Introduction and statement of results

Recently many results from the classical approximation theory have been extended to theorems treating approximation of multivariate functions by m-harmonic functions (see [1] and [8], for example, and the references in the papers therein which contain such results). The reason for this approach is the fact that, as null spaces of the even-order differential operator Δ^m, where Δ^m is the m-th iterate of the Laplace operator, the polyharmonic functions of order m inherit many of the properties of the univariate algebraic polynomials of odd degree $2m-1$. Various "extended" cubature formulae for approximate integration over the unit ball in $I\!R^n$ that are exact for spaces of polyharmonic functions have been obtained. We refer to [7], [2] and [4] for some results of this nature. In this paper we develop further the idea of extending the univariate monosplines to the multivariate setting, which was described in [7], in order to obtain a simple cubature formula for the unit square in $I\!R^2$. As an immediate consequence we obtain the bivariate cubature formula. Though the bivariate trapezoidal cubature formula is known, our approach allows to obtain its error estimates in the L_p norms as well as the error for the class of twice differentiable bivariate functions with the best possible constant. The method reveals the close relation between this cubature and Poisson's differential equation.

Let us recall a simple calculation which yield the trapezoidal quadrature. If $f \in C^2[0,1]$ we can apply integration by parts twice to the integral on the right below to obtain

$$\int_0^1 f(x)\, dx \;=\; \frac{1}{2}\left(f(0)+f(1)\right) - \int_0^1 \frac{1}{2}x(1-x)\; f''(x)\, dx. \tag{1.1}$$

Thus we have the familiar trapezoidal approximation for the integral $I(f)$ on the left, together with an error term, and the integral over $[0,1]$ is approximated by the mean value of the integrand on the "boundary" of $[0,1]$. On using a simple change

1991 *Mathematics Subject Classification.* Primary 65D32; Secondary 31B30.

Key words and phrases. Polyharmonic function, extended cubature formula, polyharmonic order of precision, trapezoidal cubature formula, Poisson's differential equation.

Research supported by the Brazilian foundations CNPq under Grant 300645/95-3 and FAPESP under Grant 97/06280-0.

of variables which adapts (1.1) to the intervals $[(j-1)h, jh],\ h := 1/n,\ j = 1, \ldots, n$ and summing the results we obtain the composite quadrature formula

$$\int_0^1 f(x)\,dx \approx h \left\{ \frac{1}{2}(f(0) + f(1)) + \sum_{k=1}^{n-1} f(kh) \right\} =: Q_n(f) \tag{1.2}$$

and the error term

$$I(f) - Q_n(f) = \int_0^1 K_n(x)\, f''(x)\, dx,$$

where

$$K_n(x) = -\frac{1}{2}(jh - x)(x - (j-1)h), \quad x \in [(j-1)h, jh], \quad j = 1, \ldots, n.$$

The function $K_n(x)$ is the Peano kernel of the linear functional $I - Q_n$ and is called the monospline associated with Q_n (see [3]).

In order to derive the bivariate extensions of the trapezoidal quadrature formula we follow a natural analogy between monosplines and the polyharmonic monosplines described in [7]. Note that the trapezoidal rule can be obtained by integrating the linear interpolant which matches the integrand at the end-points. Similarly, integration over the unit square $\bar{\Omega} = [0,1] \times [0,1]$ of the unique polynomial from *span*$\{1, x, y, xy\}$ which interpolates the data $f(0,0), f(0,1), f(1,0), f(1,1)$ yields

$$\mathbf{I}(f) := \int_0^1 \int_0^1 f(x,y)dxdy \approx \frac{1}{4}(f(0,0) + f(0,1) + f(1,0) + f(1,1)) =: C_1(f).$$

Denote by $v_{ij} = (ih, jh),\ \ 0 \le i, j \le n$ the vertices of the partition δ of $\bar{\Omega}$ induced by the grid lines $x = ih$ and $y = jh$. Divide the set of all the vertices V into three classes V_0, V_1 and V_2. The vertices $V_0 = \{v_{00}, v_{0n}, v_{n0}, v_{nn}\}$ are those of Ω, V_1 consists of the remaining boundary vertices and V_2 contains the "inside" ones, i.e. $V_2 = \{v_{ij}\ 0 < i, j < n\},\ \ V_1 = V \setminus \{V_0 \cup V_2\}$. Then the cubature formula

$$\mathbf{I}(f) \approx h^2 \left\{ \frac{1}{4} \sum_{v \in V_0} f(v) + \frac{1}{2} \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v) \right\} =: C_n(f).$$

is an extension of (1.2).

A simple but key observation is the fact that the monospline $K_1(x) = -\frac{1}{2}x(1-x)$ is the unique solution in $C^2[0,1]$ of the differential equation

$$\begin{aligned} y''(x) &\equiv 1 \quad \text{in } [0,1], \\ y(0) &= y(1) = 0. \end{aligned}$$

Let Γ be the boundary of the unit square Ω and denote by $M(x,y)$ the unique solution in $C^2(\Omega)$ of the following boundary value problem for Poisson's equation:

$$\Delta M(x,y) \equiv 1 \quad \text{in } \Omega, \tag{1.3}$$

$$M(x,y) = 0 \quad \text{on } \Gamma. \tag{1.4}$$

It is known [9, pp. 198, 332] that

$$\begin{aligned} M(x,y) &= \frac{1}{2}x(x-1) \\ &+ \frac{4}{\pi^3} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)\left\{\sinh((2k+1)\pi y) + \sinh((2k+1)\pi(1-y))\right\}}{(2k+1)^3 \sinh((2k+1)\pi)} \end{aligned}$$

Let us set

$$N(y) := \frac{\partial M}{\partial x}|_{x=1} = -\frac{\partial M}{\partial x}|_{x=0}.$$

Then

$$N(y) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{\{\sinh((2k+1)\pi y) + \sinh((2k+1)\pi(1-y))\}}{(2k+1)^2 \sinh((2k+1)\pi)}. \tag{1.5}$$

Because of the symmetry we have

$$N(x) := \frac{\partial M}{\partial y}|_{y=1} = -\frac{\partial M}{\partial y}|_{y=0},$$

which implies the following equivalent representation of $N(y)$:

$$N(y) = \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \tanh((2k+1)\pi/2)\sin((2k+1)\pi y). \tag{1.6}$$

Let $P(x) := x(x-1)N(x)/2$ and $P_n(x) := h^3 P(nx+1-i)$.

We define also the refinements of $M(x,y)$ and of $N(x)$ on δ:

$$M_n(x,y) := h^2 M(nx+1-i, ny+1-j), \qquad (x,y) \in [(i-1)h, ih] \times [(j-1)h, jh],$$

$$N_n(x) := hN(nx+1-i), \qquad x \in [(i-1)h, ih],$$

for any i, j with $1 \le i, j \le n$. Then obviously $M_1(x,y) \equiv M(x,y)$, $N_1(x) \equiv N(x)$ and $P_1(x) \equiv P(x)$.

Theorem 1. *Let for any $n \in I\!N$ the functions $M_n(x,y)$ and $N_n(x)$ be defined as above. Then the identity*

(1.7)

$$\begin{aligned}
\int_\Omega f(x,y)dxdy \quad &= \quad \int_0^1 f(x,0)N_n(x)dx + \int_0^1 f(x,1)N_n(x)dx \\
&\quad + \int_0^1 f(0,y)N_n(y)dy + \int_0^1 f(1,y)N_n(y)dy \\
&\quad + 2\sum_{i=1}^{n-1}\int_0^1 f(x,ih)N_n(x)dx + 2\sum_{j=1}^{n-1}\int_0^1 f(jh,y)N_n(y)dy \\
&\quad + \int_\Omega M_n(x,y)\Delta f(x,y)dxdy \\
&=: \quad C_{\Gamma,n}(f) + \int_\Omega M_n(x,y)\Delta f(x,y)dxdy
\end{aligned}$$

holds for every $f \in C^1(\bar{\Omega})$ for which the integral over Ω on the right hand side of (1.7) exists.

Note that $C_{\Gamma,n}$ is a linear method for the approximation of $\mathbf{I}(f)$ by $2n+2$ weighted integrals of f along the grid lines of the partition δ. The weight function is $N_n(x)$ on the boundary lines and $2N_n(x)$ on the remaining grid segments. It follows immediately from Theorem 1 that this method is precise for every function f which is harmonic on Ω. The errors of $C_{\Gamma,n}$ and C_n will now be discussed. In order to do this we require some additional definitions. For any $p \ge 1$ we denote by $L_p(\Omega)$ the L_p space on Ω equipped with the norm

$$\begin{aligned} \|f\|_p &:= \left(\int_\Omega |f(x)|dx\right)^{1/p}, \quad 1 \le p < \infty, \\ \|f\|_\infty &:= \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|. \end{aligned}$$

The Sobolev space is defined by

$$H_p(\Omega) := \left\{ f \in C^1(\bar{\Omega}) \;:\; \Delta f \;\text{ exists a.e. in } \Omega \;\text{ and }\; \Delta f \in L_p(\Omega) \right\}$$

and let $BH_p(\Omega)$ be the unit ball in $H_p(\Omega)$,

$$BH_p(\Omega) := \{ f \in H_p(\Omega) \;:\; \|\Delta f\|_p \le 1 \}.$$

Denote by $R_{\Gamma,n}(f) := \mathbf{I}(f) - C_{\Gamma,n}(f)$ and $R_n(f) := \mathbf{I}(f) - C_n(f)$ the error functionals of $C_{\Gamma,n}$ and C_n, respectively. Then

$$R_{p,\Gamma,n} := \sup \{ |R_{\Gamma,n}(f)| \;:\; f \in BH_p(\Omega) \}$$

is the maximal error of $C_{\Gamma,n}$ in $BH_p(\Omega)$.

In what follows, for any multiindex $\alpha = (\alpha_1, \alpha_2)$ and every sufficiently smooth function f, we denote by f^α the partial derivative $(\partial/\partial x)^{\alpha_1}(\partial/\partial y)^{\alpha_2} f$.

Theorem 2. *For any $n \in I\!N$ and $p \ge 1$ we have*

$$R_{p,\Gamma,n} = \|M_n\|_q, \tag{1.8}$$

where $1/p + 1/q = 1$.

If $n \in I\!N$ and $f(\mathbf{x})$ is such that $f \in C^1(\bar{\Omega})$ and $f^{(2,0)}, f^{(0,2)} \in C(\bar{\Omega})$, then there exist points $\mathbf{x}_1, \mathbf{x}_2 \in \bar{\Omega}$ for which

$$R_n(f) = -\frac{h^2}{12}\left\{ f^{(2,0)}(\mathbf{x}_1) + f^{(0,2)}(\mathbf{x}_2) \right\}. \tag{1.9}$$

Hence

$$|R_n(f)| \le \frac{h^2}{12}\left\{ \|f^{(2,0)}\|_\infty + \|f^{(0,2)}\|_\infty \right\}. \tag{1.10}$$

Moreover, $1/12$ is the smallest possible constant for which (1.10) holds.

We consider also a cubature formula of Hermitian type and obtain its error.

Theorem 3. *For the cubature formula*

$$\begin{aligned} \int_\Omega f(x,y)dxdy &= C_n(f) \\ &\quad + \frac{h^3}{48}\left(-f^{(1,0)}(0,0) - f^{(0,1)}(0,0) + f^{(1,0)}(1,0) - f^{(0,1)}(1,0) \right) \\ &\quad + \frac{h^3}{48}\left(-f^{(1,0)}(0,1) + f^{(0,1)}(0,1) + f^{(1,0)}(1,1) + f^{(0,1)}(1,1) \right) \\ &\quad + \frac{h^3}{24}\sum_{k=1}^{n-1}\left(-f^{(1,0)}(0,kh) + f^{(1,0)}(1,kh) \right) \\ &\quad + \frac{h^3}{24}\sum_{k=1}^{n-1}\left(f^{(0,1)}(kh,0) - f^{(0,1)}(kh,1) \right) \\ &=: C_{n,H}(f), \end{aligned} \tag{1.11}$$

the error estimate

$$|\mathbf{I}(f) - C_{n,H}(f)| \leq \frac{h^2}{24}\left(\|f^{(2,0)}\|_\infty + \|f^{(0,2)}\|_\infty\right) + \frac{h^3}{60}\left(\|f^{(2,1)}\|_\infty + \|f^{(1,2)}\|_\infty\right)$$

holds for every $f \in C^3(\bar{\Omega})$.

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let us recall that the first Green's formula reads as

$$\int_\Gamma \left(u\frac{\partial}{\partial\nu}v - v\frac{\partial}{\partial\nu}u\right) d\sigma \;+\; \int_\Omega (u\Delta v - v\Delta u)\, dxdy \;=\; 0, \tag{2.1}$$

where $\frac{\partial}{\partial\nu}$ is the inner normal derivative and $d\sigma$ is the element of the intrinsic measure σ on Γ. It holds for every $u, v \in C^1(\bar{\Omega})$ for which the above integral over Ω exists. Applying (2.1) to $u = f$ and $v = M$ and taking into account (1.3) and (1.4) we obtain

$$\int_\Omega f(x,y)dxdy = -\int_\Gamma f\frac{\partial}{\partial\nu}M d\sigma + \int_\Omega M(x,y)\Delta f(x,y)dxdy.$$

Since ν is the intrinsic normal to Γ we have

$$\begin{aligned}
\frac{\partial}{\partial\nu}M(x,0) &= \tfrac{\partial M}{\partial y}|_{y=0} = & -N(x),\\
\frac{\partial}{\partial\nu}M(x,1) &= -\tfrac{\partial M}{\partial y}|_{y=1} = & -N(x),\\
\frac{\partial}{\partial\nu}M(0,y) &= \tfrac{\partial M}{\partial x}|_{x=0} = & -N(y),\\
\frac{\partial}{\partial\nu}M(0,y) &= -\tfrac{\partial M}{\partial x}|_{x=1} = & -N(y).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_\Omega f(x,y)dxdy \;\;&=\;\; \int_2^1 f(x,0)N(x)dx + \int_0^1 f(x,1)N(x)dx\\
&\quad + \int_0^1 f(0,y)N(y)dy + \int_0^1 f(1,y)N(y)dy\\
&\quad + \int_\Omega M_n(x,y)\Delta f(x,y)dxdy.
\end{aligned}$$

An appropriate change of variables transforms the latter to each cell of the partition δ. The summation of the resulting equalities yields (1.7). $\square$

Lemma 1. *For any* $x \in (0,1)$ *and* $y \in (0,1)$

$$M(x,y) < 0$$

and

$$N(x) > 0, \quad N(y) > 0. \tag{2.2}$$

Moreover $N(x)$ *is continuous on* $[0,1]$ *and is even with respect to* $x = 1/2$.

Proof. We need some basic facts about subharmonic functions. The function v is subharmonic in Ω if for any circle $B \subset \Omega$ the solution U of the Dirichlet's problem $\Delta U = 0$ on B, $U|_{\partial B} = v|_{\partial B}$, satisfies the inequality $v < U$ on B. It is well known that $v \in C^2(\Omega)$ is subharmonic in Ω if and only if $\Delta v > 0$ there. If $v \in C(\bar{\Omega})$ is subharmonic and U is harmonic in Ω such that $U \geq v$ on $\partial\Omega$ then $U > v$ in Ω.

Since $\Delta M \equiv 1$ on Ω then M is subharmonic. Moreover $M \in C(\bar{\Omega})$ and $M|_\Gamma = 0$. Therefore $M(x,y) < 0$ in Ω.

In order to prove that $N(x)$ is continuous and is even with respect to $x = 1/2$, recall that

$$N(x) = 1/2 - T(x),$$

where

$$T(x) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{\{\sinh((2k+1)\pi x) + \sinh((2k+1)\pi(1-x))\}}{(2k+1)^2 \sinh((2k+1)\pi)}. \tag{2.3}$$

It is easy to see that for any $k \in I\!N$ the function $\sinh((2k+1)\pi x) + \sinh((2k+1)\pi(1-x))$ is positive on $(0,1)$, is convex and is even with respect to $x = 1/2$. Then for every $x \in (0,1)$ we have

$$0 < \sinh((2k+1)\pi x) + \sinh((2k+1)\pi(1-x)) < \sinh((2k+1)\pi).$$

Thus the series on the right hand side of (2.3) is absolutely and uniformly convergent. Hence T is a continuous function. Furthermore the latter inequality and the equality $\sum_{k=0}^{\infty} 1/(2k+1)^2 = \pi^2/8$ yield $T(x) < 1/2$, which is equivalent to (2.2). □

Corollary 1. *The function P is continuous and negative on $(0,1)$ and is even with respect to $1/2$.*

Lemma 2. *We have*

$$\int_0^1 N(x)dx = \frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{\tanh((2k+1)\pi/2)}{(2k+1)^3} = \frac{1}{4} \tag{2.4}$$

and

$$\int_0^1 xN(x)dx = \frac{1}{8}$$

Proof. We proved in Lemma 1 that the series which represent $N(x)$ are uniformly convergent so that $N(x)$ can be integrated termwise. It is easily seen that

$$\int_0^1 \sin((2k+1)\pi x)dx = \frac{2}{(2k+1)\pi}.$$

Then the second expression (1.6) for $N(x)$ yields

$$\int_0^1 N(x)dx = \frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{\tanh((2k+1)\pi/2)}{(2k+1)^3}. \tag{2.5}$$

Similarly, on using the fact that

$$\int_0^1 \sinh((2k+1)\pi x) + \sinh((2k+1)\pi(1-x))dx = \frac{2}{(2k+1)\pi}(\cosh((2k+1)\pi) - 1)$$

and the first representation (1.5) of $N(x)$, we obtain

$$\int_0^1 N(x)dx = \frac{1}{2} - \frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{\tanh((2k+1)\pi/2)}{(2k+1)^3}. \tag{2.6}$$

Now the average of (2.5) and (2.6) implies (2.4). It is also easily seen by integration of $x\sin((2k+1)\pi x)$ that

$$\int_0^1 xN(x)dx = \frac{4}{\pi^3}\sum_{k=0}^{\infty}\frac{\tanh((2k+1)\pi/2)}{(2k+1)^3}$$

which is exactly the half of the integral of $N(x)$. $\square$

Now we shall obtain C_n from $C_{\Gamma,n}$. For this purpose the trapezoidal quadrature formula for weighted integrals will be applied to the integrals which appear in $C_{\Gamma,n}$. The method is demonstrated first on the interval $y = 0, 0 \le x \le h$. Let $\tilde{N}(x) = N_n(x)$ for $x \in [0,h]$. Then

$$\tilde{N}(x) = \frac{4h}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\tanh((2k+1)\pi/2)\ \sin((2k+1)\pi nx).$$

The integrand $f(x,0)$ is approximated on $[0,h]$ by its Lagrange interpolating polynomial at 0 and h:

$$f(x,0) = f(0,0)(1-nx) + f(h,0)nx + f[0,h,x;0]x(x-h),$$

where $f[0,h,x;0]$ is the divided difference of f at $(0,0)$, $(h,0)$ and $(x,0)$. Multiplying the latter by $\tilde{N}(x)$ and integrating over $[0,h]$ we obtain

$$\begin{aligned}\int_0^h f(x,0)N_n(x)dx \quad &= \quad \int_0^h \tilde{N}(x)(1-nx)dx\ f(0,0) + \int_8^h \tilde{N}(x)nxdx\ f(h,0)\\ &\quad + \int_0^h f[0,h,x;0]\tilde{N}(x)x(x-h)dx.\end{aligned}$$

Lemma 2 implies

$$\int_0^h \tilde{N}(x)(1-nx)dx = \int_0^h \tilde{N}(x)nxdx = h^2/8.$$

We obtained the formula

$$\int_0^h f(x,0)N_n(x)dx = \frac{h^2}{8}\{f(0,0)+f(h,0)\} + R(h;x;f).$$

Proof of Theorem 2 It follows immediately from Theorem 1 that

$$R_{\Gamma,n}(f) = \int_\Omega M_n(x,y)\Delta f(x,y)dxdy.$$

On applying Hölder's inequality we obtain

$$|R_{\Gamma,n}(f)| = \|M_n\|_q\|\Delta f\|_p.$$

If $f \in BH_p(\Omega)$ then $|R_{\Gamma,n}(f)| \le \|M_n\|_q$. Hence $|R_{p,\Gamma,n}(f)| \le \|M_n\|_q$. Let $1 < p \le \infty$. Equality (6) will be proved if we indicate a function $f \in BH_p(\Omega)$ for which $|R_{\Gamma,n}(f)| = \|M_n\|_q$. Every function $f \in H_p$, such that

$$\Delta f(x) = \left(\int_B |M(x)|^q\ dx\right)^{-1/p} |M(x)|^{q-1}\ \mathrm{sign}M(x)$$

belongs to $BH_p(\Omega)$ and obviously $|R_{\Gamma,n}(f)| = \|M\|_q\|\Delta^2 f\|_p$. For $p = 1$ we let q tend to infinity in the equality $|R_{p,\Gamma,n}| = \|M_n\|_q$. This yields $|R_{1,\Gamma,n}| = \|M_n\|_\infty$.

Bearing in mind that, if $x \in [0,h]$, then $2f[0,h,x;0] = f^{(2,0)}(\xi,0)$ for some $\xi \in (0,h)$ we conclude that, for every f for which $f^{(2,0)} \in C(\bar{\Omega})$,

$$|R(h;x;f)| \le \|f^{(2,0)}\|_\infty \int_0^h |P_n(x)|dx.$$

On applying the same approach to all the edges of the partition δ we obtain C_n and the error estimate

$$|C_{\Gamma,n}(f) - C_n(f)| \le 2n\|f^{(2,0)}\|_\infty \int_0^1 |P_n(x)|dx + 2n\|f^{(0,2)}\|_\infty \int_0^1 |P_n(y)|dy.$$

The simple observation that $\|\Delta f\|_\infty \le \|f^{(2,0)}\|_\infty + \|f^{(0,2)}\|_\infty$ yields the error estimate

$$|R_n(f)| \le \left\{2n \int_0^1 |P_n(x)|dx + \|M_n\|_1\right\} \left\{\|f^{(2,0)}\|_\infty + \|f^{(0,2)}\|_\infty\right\}. \tag{2.7}$$

In view of (2.7), for the proof of (1.10) we need only to find the L_1 norms of P_n and M_n. It follows from their definitions, (2.2) and Corollary 1 that

$$\int_0^1 |P_n(x)|dx = -h^3 \int_0^1 P(x)dx \tag{2.8}$$

and

$$\|M_n\|_1 = -h^2 \int_\Omega M(x,y)dxdy. \tag{2.9}$$

Since the series which participate in the representations of P and M are uniformly convergent then the integrals on the right hand side of (2.8) and (2.9) can be calculated by termwise integration of these series. We omit the technical details. The results are

$$\int_0^1 P(x)dx = -\frac{8}{\pi^5} \sum_{k=0}^\infty \frac{\tanh((2k+1)\pi/2)}{(2k+1)^5}$$

and

$$\int_\Omega M(x,y)dxdy = -\frac{1}{12} + \frac{16}{\pi^5} \sum_{k=0}^\infty \frac{\tanh((2k+1)\pi/2)}{(2k+1)^5}.$$

On using (2.8) and (2.9) we obtain

$$\begin{aligned} 2n \int_0^1 |P_n(x)|dx + \|M_n\|_1 &= 2nh^3 \frac{8}{\pi^5} \sum_{k=0}^\infty \frac{\tanh((2k+1)\pi/2)}{(2k+1)^5} \\ &\quad + h^2 \left\{ \frac{1}{12} - \frac{16}{\pi^5} \sum_{k=0}^\infty \frac{\tanh((2k+1)\pi/2)}{(2k+1)^5} \right\} \\ &= \frac{1}{12} h^2. \end{aligned}$$

In order to prove the sharpness of the constant $1/12$ we observe that the error estimate (1.10) is attained for the simple function $g(x,y) = x^2$, namely we have

$$|R_n(g)| = h^2/12 \left\{\|g^{(2,0)}\|_\infty + \|g^{(0,2)}\|_\infty\right\}. \tag{2.10}$$

Indeed, obviously $\mathbf{I}(g) = 1/3$ and simple calculations show that $C_n(g) = 1/3 + h^2/6$. Hence $|R_n(g)| = h^2/6$ which is equivalent to (2.10). $\square$

We conclude this section with some remarks. There are some other ways to obtain error estimates for C_n. The first of these is to pass through uniform bounds. Recall the method used to obtain C_n in the introduction. Let $s_n(f;x,y)$ be the unique piecewise continuous spline which reduces to *span*$\{1,x,y,xy\}$ on every cell and interpolates f at the vertices of δ. Then $C_n(f) = \int_\Omega s_n(f;x,y)dxdy$. On the other hand, employing the quasi-interpolation technique [5] we conclude that $\|f - s_n(f)\|_\infty = O(h^2)$. This gives $|R_n(f)| = O(h^2)$. The double reduction approach provides the best constant. Finally we mention that just like $Q_n(f) \longrightarrow I(f)$ for every Riemann integrable function $f(x)$ on $[0,1]$, $C_n(f) \longrightarrow \mathbf{I}(f)$ for every Riemann integrable function $f(x,y)$ on $\bar{\Omega}$. The proofs are very simple. Recall that for the proof of the convergence of the quadrature formula one only observes that $Q_n(f)$ is a Riemann sum of $f(x)$ on the partition δ_Q of $[0,1]$ induced by the points $(1/2+i)h, i = 0,\ldots,n-1$. Similarly $C_n(f)$ is a Riemann sum of $f(x,y)$ on the partition of $\bar{\Omega}$ which is the Descartes product of δ_Q.

3. Hermitian type cubature formulae

Proof of Theorem 3. In what follows we suppose that

$$M(x,y) = (1/4)(x(x-1) + y(y-1)).$$

Since $\Delta M(x,y) \equiv 1$, on applying Green's formula (2.1) to the refinement $M_n(x,y) = (1/4)(x(x-h+y(y-h)$ of M and to f on the square $[0,h]\times[0,h]$ of the partition δ, we obtain

$$\begin{aligned}
\int_0^h \int_0^h f(x,y)dxdy &= -\int_\Gamma \frac{\partial M}{\partial \nu} f\, d\sigma(\xi) + \int_\Gamma \frac{\partial f}{\partial \nu} M\, d\sigma(\xi) + \int_\Omega M \Delta f dxdy \\
&= \frac{h}{4}\left(\int_0^h f(0,y)dy + \int_0^h f(x,0)dx\right) \\
&\quad + \frac{h}{4}\left(\int_0^h f(h,y)dy + \int_0^h f(x,h)dx\right) \\
&\quad + \int_0^h \frac{y(y-h)}{4} f^{(1,0)}(0,y)dy - \int_0^h \frac{y(y-h)}{4} f^{(1,0)}(h,y)dy \\
&\quad + \int_0^h \frac{x(x-h)}{4} f^{(0,1)}(x,0)dx - \int_0^h \frac{x(x-h)}{4} f^{(0,1)}(x,h)dx \\
&\quad + \int_\Omega M(x,y)\Delta f(x,y)dxdy.
\end{aligned}$$

Now we use the representation

$$f(x) = (h-x)/hf(0) + x/hf(h) + f[0,h,x]x(x-h)$$

in order to approximate the first eight integrals in the above representation by the trapezoidal quadrature formula. Bearing in mind that for any $x \in [0,h]$ there is

$\eta \in [0,h]$, such that $f[0,h,x] = f''(\eta)$, we obtain

$$\begin{aligned}
\int_0^h \int_0^h f(x,y)dxdy &= \frac{h^2}{4}\left(f(0,0)+f(h,0)+f(0,h)+f(h,h)\right) \\
&-\frac{h^4}{48}\left(f^{(2,0)}(\eta_1,0)+f^{(2,0)}(\eta_2,h)+f^{(0,2)}(0,\eta_3)\right) \\
&-\frac{h^4}{48}\left(f^{(0,2)}(h,\eta_4)+f^{(1,0)}(0,0)+f^{(0,1)}(0,0)\right) \\
&+\frac{h^3}{48}\left(f^{(1,0)}(h,0)+f^{(0,1)}(0,h)+f^{(1,0)}(h,h)\right) \\
&+\frac{h^3}{48}\left(f^{(0,1)}(h,h)-f^{(0,1)}(h,0)-f^{(1,0)}(0,h)\right) \\
&+\frac{h^5}{120}\left(f^{(1,2)}(0,\eta_5)-f^{(1,2)}(h,\eta_6)\right) \\
&+\frac{h^5}{120}\left(f^{(2,1)}(\eta_7,0)-f^{(2,1)}(\eta_8,h)\right) \\
&-\frac{h^4}{12}\Delta f(\eta_9,\eta_{10}),
\end{aligned}$$

where $\eta_j \in [0,h]$ for $j = 1,\ldots,10$. We sum the corresponding integrals over the cells of δ and take into account that the derivatives of order two and three are continuous in Ω. The result is

$$\int_\Omega f(x,y)dxdy = C_{n,H}(f) + R_{n,H}(f),$$

with

$$\begin{aligned}
R_{n,H}(f) &= -\frac{h^2}{24}\left(f^{(2,0)}(\xi_1,\xi_2)+f^{(0,2)}(\xi_3,\xi_4)\right) \\
&+\frac{h^3}{120}\left(f^{(1,2)}(\xi_5,\xi_6)-f^{(1,2)}(\xi_7,\xi_8)+f^{(2,1)}(\xi_9,\xi_{10})-f^{(2,1)}(\xi_{11},\xi_{12})\right) \\
&-\frac{h^2}{12}\Delta f(\xi_{13},\xi_{14}),
\end{aligned}$$

where $\xi_j \in [0,1]$ for $j = 1,\ldots,14$. Therefore

$$|R_{n,H}(f)| \le \frac{h^2}{24}\left(\|f^{(2,0)}\|_\infty + \|f^{(0,2)}\|_\infty\right) + \frac{h^3}{60}\left(\|f^{(2,1)}\|_\infty + \|f^{(1,2)}\|_\infty\right).$$

□

The proof of Theorem 3 shows once again that the relation between cubature formulae and polyharmonic monosplines, as developed in [7] generates various interesting cubature formulae even if the monospline is relatively simple. We conclude the paper with an example of a cubature formula of Hermitian type we consider very natural. It was proved in [6] that, for any square and for any smooth function $f(x,y)$, there exists a unique bivariate polynomial $p(f;x,y)$ from

$$\operatorname{span}\{1,x,y,x^2,xy,y^2,x^3,x^2y,xy^2,y^3,x^3y,xy^3\},$$

such that, for the four vertices $V_k,\ k = 1,2,3,4$, of the square

$$p^{(i,j)}(f;V_k) = f^{(i,j)}(V_k) \qquad 0 \le i+j \le 1.$$

An integration of the interpolating polynomial $p(f;x,y)$ over the squares of δ yields the cubature formula

$$\begin{aligned}\int_\Omega f(x,y)dxdy &= C_n(f)\\ &+\frac{h^3}{24}\sum_{k=0}^{n}\Big(f^{(1,0)}(0,kh)-f^{(1,0)}(1,kh)\Big)\\ &+\frac{h^3}{24}\sum_{k=0}^{n}\Big(f^{(0,1)}(kh,0)-f^{(0,1)}(kh,1).\Big)\end{aligned}$$

The general theory of quasi interpolants yields that its error must be a linear combination of the norms of $f^{(4,0)}$, $f^{(2,2)}$ and $f^{(0,4)}$. The problem of determining the best possible constants is of interest. We shall return to this question elsewhere.

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