NON-NEGATIVE TRIGONOMETRIC POLYNOMIALS

DIMITAR K. DIMITROV AND CLINTON A. MERLO

Abstract

An extremal problem for the coefficients of sine polynomials, which are non-negative in $[0,\pi]$, posed and discussed by Rogosinski and Szegő is under consideration. An analog of the Fejér-Riesz representation of non-negative general trigonometric and cosine polynomials is proved for non-negative sine polynomials. Various non-negative sine and cosine polynomials are obtained in an explicit form.

1. *Introduction*

One of the earliest examples of non-negative trigonometric series is the Poisson kernel

$$1+2\sum_{k=1}^{\infty}\rho^k\cos k\theta=\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2},\quad -1<\rho<1,$$

which, as it was pointed out by Askey and Gasper [**3**], was also found but not published by Gauss [**14**]. The problem of constructing non-negative trigonometric polynomials was inspired by the development of the theory of Fourier series and this area of research reached its efflorescence in the beginning of this century due to the efforts of some celebrated mathematicians.

It is well known that, for every 2π-periodic function $f\in L_p[-\pi,\pi]$, $1\le p\le\infty$, the sequence of convolutions $k_n * f$ converges to f in the $L_p[-\pi,\pi]$-norm provided k_n are positive summability kernels [**23**, p. 9]. Summability kernels are even positive trigonometric polynomials which approximate Dirac's delta $\delta(\theta)$. It seems Fejér was the first to realise this fact. He proved in [**8**] that

$$K_n(\theta)=1+2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\cos k\theta \tag{1.1}$$

are non-negative trigonometric polynomial approximations of the formal Fourier series

$$1+2\sum_{k=1}^{\infty}\cos k\theta \tag{1.2}$$

of $\delta(\theta)$ and established the uniform convergence of the sequence $(1/2\pi)\,K_n * f$ to f for any continuous 2π-periodic function f. Later on Jackson [**21**, **22**] proved that,

---

Research supported by the Brazilian Science Foundations CNPq under Grant 300645/95-3, FAPESP under Grants 97/06280-0 and 98/11977-3.

1991 *Mathematics Subject Classification* 42A05, 26D05.

for the kernel

$$J_n(\theta) = \frac{1}{\gamma_n}\left(\frac{\sin(n\theta/2)}{n\sin(\theta/2)}\right)^4, \quad \gamma_n = \int_{-\pi}^{\pi}\left(\frac{\sin(n\theta/2)}{n\sin(\theta/2)}\right)^4 dt,$$

the rate of uniform convergence of $J_n * f$ to any continuous 2π-periodic function f is $\omega(f;1/n)$, where $\omega(f;\delta)$ denotes the modulus of continuity of f (see [**36**, Chapter X, §6]). This is the classical Jackson's theorem in approximation theory [**26**, **24**]. It is worth mentioning that $\omega(f;1/n)$ is the best possible rate of uniform approximation of continuous functions by trigonometric polynomials of order n (see [**30**, Chapter 6]).

Another reason Fejér was interested in non-negative trigonometric polynomials is the Gibbs' phenomenon [**36**, Chapter II, §9]. In 1910 Fejér conjectured that the partial sums

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k} \tag{1.3}$$

of the sine Fourier series of the function $(\pi-\theta)/2,\ 0<\theta<\pi$, are positive in $(0,\pi)$. Fejér's conjecture was proved independently and almost simultaneously by Jackson [**20**] and Gronwall [**17**]. See [**18**] and [**25**, Chapters 2.2 and 4] for interesting surveys on Gibbs' phenomenon and for the Fejér-Jackson-Gronwall inequality. W. H. Young [**35**] proved that the cosine polynomials $1+\sum_{k=1}^{n}(\cos k\theta)/k$ are also non-negative and this result was extended by Rogosinski and Szegő [**27**] and by Gasper [**13**].

There are still other reasons for the interest in positive trigonometric polynomials and one of these is their close relation to univalent functions and polynomials. Alexander [**1**] proved that the polynomial $\sum_{k=1}^{n} a_k z^k/k$ is univalent in the unit disc of the complex plane provided $a_1 \geq \cdots \geq a_n > 0$. Dieudoneé [**6**] revealed the connection between Alexander's result and Fejér-Jackson-Gronwall's inequality. Dieudoneé proved that the polynomial $\sum_{k=1}^{n} a_k z^k$ is univalent in the unit disc if and only if

$$\sum_{k=1}^{n} a_k z^{k-1}\frac{\sin k\theta}{\sin\theta} \neq 0 \quad \text{for} \quad |z|<1,\ 0\leq\theta\leq\pi.$$

Later contributions on this connection are due to Fejér [**10**, **11**] and Fejér and Szegő [**12**] who proved that an analytic function $f(z)=u(z)+iv(z)$ with real Maclaurin coefficients is univalent in the unit disc if $\partial u(e^{i\theta})/\partial\theta \leq 0$ for $0\leq\theta\leq\pi$. Functions which obey the latter property are called "vertically convex". We refer to the recent survey of Gluchoff and Hartmann [**15**] for these and other interesting results on the interplay of univalent functions and non-negative trigonometric series.

The classical non-negative trigonometric polynomials inspired interesting generalisations to positive Jacobi polynomial sums. One of the best-known results in this direction is the inequality of Askey and Gasper [**2**]. It played an important role in the final step of de Brange's proof [**5**] of the Bieberbach conjecture. Askey and Gasper's survey [**3**] is a good source for information about positive Jacobi polynomial sums.

One more immediate application of the non-negative trigonometric polynomials is to the theory of orthogonal polynomials on the unit circle. A general class of such orthogonal polynomials are the so-called Bernstein-Szegő polynomials. They are orthogonal with respect to an absolutely continuous measure $d\nu(\theta)$ whose Radon-Nikodym derivative $d\nu(\theta)/d\theta$ is a reciprocal of a positive trigonometric polynomial (see [**34**]). This kind of measures are the simplest examples of measures which belong to Szegő's class. The corresponding orthogonal polynomials have an interesting application to prediction of time series [**16**].

The basic tool in the proof of the orthogonality of Bernstein-Szegő polynomials is the Fejér and Riesz (see [**9**]) representation of non-negative trigonometric polynomials: For every non-negative trigonometric polynomial $T(\theta)$,

$$T(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta),$$

there exists an algebraic polynomial $R(z) = \sum_{k=0}^{n} c_k z^k$ of degree n such that $T(\theta) = |R(e^{i\theta})|^2$. Conversely, for every algebraic polynomial $R(z)$ of degree n, the polynomial $|R(e^{i\theta})|^2$ is a non-negative trigonometric polynomial of order n.

Straightforward calculations yield the following equivalent form of this result:

Theorem 1. *The trigonometric polynomial*

$$T(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta)$$

of order n is non-negative for every real θ if and only if there exist complex numbers c_k, $k = 0, 1, \dots, n$, such that

$$a_0 = \sum_{k=0}^{n} |c_k|^2, \quad a_k - ib_k = 2 \sum_{\nu=0}^{n-k} c_{k+\nu} \bar{c}_\nu, \quad \text{for} \quad k = 1, \dots, n.$$

This form of Fejér-Riesz' representation shows that a variety of non-negative trigonometric polynomials can be obtained by setting different values for the parameters c_k. However, the most natural and beautiful examples arise as solutions of some extremal problems for the coefficients of $T(\theta)$. Fejér [**9**] determined the maximum of $\sqrt{a_1^2 + b_1^2}$ subject $T(\theta)$ is non-negative . He showed that

$$\sqrt{a_1^2 + b_1^2} \le 2 \cos(\pi/(n+2)) \tag{1.4}$$

and that this bound is sharp. A nice simple proof of Jackson's approximation theorem in Rivlin [**26**, Chapter I] uses essentially Fejér's inequality (1.4). It is easily seen that Fejér's kernel (1.1) is the only solutions of the extremal problem

$$\max\{a_1 + \cdots + a_n \ : \ 1 + 2\sum_{k=1}^{n} a_k \cos k \ge 0\}.$$

We shall prove that the sine polynomial

$$\sum_{k=0}^{m} \sin(2k+1)\theta = \frac{\sin^2((m+1)\theta)}{\sin\theta} \tag{1.5}$$

turns out to be the only sine polynomial of the form $\sum_{k=1}^{2m+1} b_k \sin k\theta$, $b_1 = 1$, that is non-negative in $[0, \pi]$, for which the maximum of the moment b_n, $n = 2m+1$, is attained.

Szegő [**29**] and Egerváry and Szász [**7**] extended Fejér's result (1.4), finding estimates for the means $\sqrt{a_k^2 + b_k^2}$, for $k = 1, 2, \dots, n$, of non-negative trigonometric polynomials. Their result reads as follows:

THEOREM 2. *If*

$$T(\theta) = 1 + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta)$$

is non-negative , then for any $k,\ 1 \leq k \leq n$,

$$\sqrt{a_k^2 + b_k^2} \leq 2 \cos \frac{\pi}{[n/k] + 2},$$

where $[x]$ *denotes the integer part of* x. *Moreover, equality is attained if and only if*

$$T(\theta) = \tau(\theta) \left\{ 1 + \frac{2}{p+2} \sum_{\nu=1}^{p} \left((p - \nu + 1) \cos \nu\alpha + \frac{\sin(\nu+1)\alpha}{\sin \alpha} \right) \cos(\nu m(\theta - \psi)) \right\},$$

where τ *is an arbitrary non-negative trigonometric polynomial of order* q, $\alpha = \pi/(p+2)$, $p = [n/k]$, $n = pk + q$, $(0 \leq q < k)$ *and* ψ *is an arbitrary constant.*

In particular, one obtains, for any $k,\ 1 \leq k \leq n$, the sharp estimates

$$|a_k| \leq \cos \frac{\pi}{[n/k] + 2}$$

for the coefficients of the non-negative cosine polynomial

$$1/2 + \sum_{k=1}^{n} a_k \cos k\theta.$$

Since sine polynomials are odd functions, in what follows we shall call

$$S_n(\theta) = \sum_{k=1}^{n} b_k \ \sin k\theta$$

a non-negative sine polynomial if $S_n(\theta) \geq 0$ for every $\theta \in [0, \pi]$. It is clear that, if $S_n(\theta)$ is non-negative, then $b_1 \geq 0$ and $b_1 = 0$ if and only if S_n is identically zero. Motivated by the results for general trigonometric and for cosine polynomials described above and having in mind this observation about b_1, Rogosinski and Szegő [**28**] considered the following extremal problem for non-negative sine polynomials:

Determine the minimum and maximum values of b_k *provided that the sine polynomial* $S_n(\theta)$ *is in*

$$\mathcal{S}_n = \left\{ S_n(\theta) = \sin\theta + \sum_{k=2}^{n} b_k \ \sin k\theta \quad : \quad S_n(\theta) \geq 0 \quad \text{for} \quad \theta \in [0, \pi] \right\}.$$

For each of these values, find the extremal sine polynomial which belongs to $\mathcal{S}_n$ *and whose coefficient* b_k *coincides with the corresponding extremal value.*

Although Rogosinski and Szegő suggested two methods for obtaining the extrema of the moments b_k, they found the minimal and maximal values only for b_2, b_3, b_{n-1} and b_n. Moreover, in none of the cases the corresponding extremal sine polynomials were determined explicitly.

In this paper we develop a method of solving the above stated problem. The basic tool is a general representation of non-negative sine polynomials. Note that Theorem 1 implies a representation of non-negative cosine polynomials (see Lemma 2 below). However, to the best of our knowledge, no similar representation of non-negative sine polynomial was known. Due to the complicated nature of the problem, we apply the method to find the extremal polynomials, associated with the minima and maxima of b_n, b_{n-1} and b_{n-2}. The proposed method has the advantage that every non-negative sine polynomial we obtain yields a corresponding non-negative cosine polynomial. Another feature of the method is it is purely algorithmic. Thus a computer program would generate the extremal values of the moments b_k together with the corresponding extremal non-negative sine polynomials.

The results containing a variety of non-negative sine and cosine polynomials are stated in the next section. We formulate and prove only some of the possible consequences of the main results. We do not include, for example, the non-negative trigonometric polynomials which are in the convex hulls of the polynomials that we obtain. The remark which follows the statements of Vietoris' [**33**] results in Askey and Steinig's paper [**4**] gives the idea how new non-negative trigonometric polynomials can be generated by a known sequence of non-negative trigonometric polynomials. Our method is developed in Section 3. The extremal problems for b_{n-1} and for b_n are solved in Section 4. In Section 5 the minimal and the maximal values of b_{n-2} are determined. For odd n, the corresponding extremal polynomials are constructed. The limit behaviour of the extremal values of b_{n-2}, for even n, is established.

2. *Statement of results.*

THEOREM 3. *Let $n = 2m+2$ be a even positive integer. Then for every $S_n(\theta) \in \mathcal{S}_n$*

(i)

$$-\frac{m+1}{m+2} \le b_n \le \frac{m+1}{m+2}.$$

The equality $b_n = (m+1)/(m+2)$ is attained only for the non-negative sine polynomial

$$\sum_{k=0}^{m} \left(\frac{(m+k+2)(m-k+1)}{(m+1)(m+2)} \sin(2k+1)\theta + \frac{(k+1)^2}{(m+1)(m+2)} \sin(2k+2)\theta \right). \tag{2.1}$$

The equality $b_n = -(m+1)/(m+2)$ is attained only for the non-negative sine polynomial

$$\sum_{k=0}^{m} \left(\frac{(m+k+2)(m-k+1)}{(m+1)(m+2)} \sin(2k+1)\theta - \frac{(k+1)^2}{(m+1)(m+2)} \sin(2k+2)\theta \right). \tag{2.2}$$

(ii)

$$-\frac{m}{m+2} \le b_{n-1} \le 1.$$

The equality $b_{n-1} = 1$ is attained only for the non-negative sine polynomials

$$\sin\theta + \sum_{k=1}^{m} (2pq \sin 2k\theta + \sin(2k+1)\theta) + pq \sin(2m+2)\theta, \tag{2.3}$$

where the parameters p and q satisfy $p^2 + q^2 = 1$.
The equality $b_{n-1} = -m/(m+2)$ is attained only for the non-negative sine polyno-

mials

$$\begin{aligned} \sin\theta &+ \sum_{k=1}^{m}\left\{2pq\left(1-\frac{2k^2}{m(m+2)}\right)\sin 2k\theta+\left(1-\frac{2k(k+1)}{m(m+2)}\right)\sin(2k+1)\theta\right\} \\ &- pq\sin(2m+2)\theta, \end{aligned} \tag{2.4}$$

where p and q satisfy the relation $p^2+q^2=1$.

An interesting non-negative sine polynomial is

$$\sum_{k=0}^{m}\frac{(m+k+2)(m-k+1)}{(m+1)(m+2)}\sin(2k+1)\theta,$$

which is a convex combination of (2.1) and (2.2). Observe that the extremal polynomials (2.3) and (2.4) depend on a parameter. Their convex combination also has an intriguing form.

THEOREM 4. *Let $n=2m+1$ be an odd positive integer. Then for every $S_n(\theta)\in\mathcal{S}_n$*

(i)

$$\frac{-m}{m+2}\le b_n\le 1.$$

The equality $b_n=1$ is attained only for the non-negative sine polynomial (1.5). The equality $b_n=-m/(m+2)$ is attained only for

$$\sum_{k=0}^{m}\left((1-\frac{2k(k+1)}{m(m+2)})\sin(2k+1)\theta\right). \tag{2.5}$$

(ii)

$$-1\le\ b_{n-1}\le 1.$$

The equality $b_{n-1}=1$ is attained only for the non-negative sine polynomial

$$\sum_{k=1}^{2m}\sin k\theta+\frac{1}{2}\sin(2m+1)\theta. \tag{2.6}$$

The equality $b_{n-1}=-1$ is attained only for the non-negative sine polynomial

$$\sum_{k=1}^{2m}(-1)^{k+1}\sin k\theta+\frac{1}{2}\sin(2m+1)\theta. \tag{2.7}$$

It is worth noting the extremal property of the sine polynomial (1.5). Another interesting observation is that the polynomial (2.5) coincides with the polynomial (2.4) if $p=0$ or $q=0$.

THEOREM 5. *Let $n=2m+1$, $m\ge 2$ be an odd positive integer. Then for every $S_n(\theta)\in\mathcal{S}_n$*

$$\frac{-m+4-\sqrt{m(5m-8)}}{2(m+2)}\le b_{n-2}\le\frac{1+\sqrt{5}}{2}.$$

The equality $b_{n-2}=(1+\sqrt{5})/2$ is attained only for the non-negative sine polynomial

$$\begin{aligned}\sin\theta \quad &+ \quad \sum_{k=1}^{m-2}\left(\frac{5+2\sqrt{5}}{5}\right)\sin(2k+1)\theta \\ &+ \quad \left(\frac{1+\sqrt{5}}{2}\right)\sin(2m-1)\theta+\left(\frac{5+\sqrt{5}}{10}\right)\sin(2m+1)\theta.\end{aligned} \tag{2.8}$$

The equality $b_{n-2}=(-m+4-\sqrt{m(5m-8)})/(2m+4)$ *is attained only for the non-negative sine polynomial*

$$\begin{aligned}\sin\theta \quad &+ \quad \sum_{k=1}^{m-2}\left(\frac{4+2k+2k^2-m^2-\frac{2(2k(k+1)-2(k^2+k+2)m+m^3)}{\sqrt{m(5m-8)}}}{m^2-4}\right)\sin(2k+1)\theta \\ &+ \quad \left(\frac{4-m-\sqrt{m(5m-8)}}{2(m+2)}\right)\sin(2m-1)\theta \\ &- \quad \left(\frac{m+\frac{(m-4)m}{\sqrt{m(5m-8)}}}{2(m+2)}\right)\sin(2m+1)\theta.\end{aligned} \tag{2.9}$$

In order to formulate our result concerning the extremal values of b_{n-2} for the case where n is even we need the following technical result:

LEMMA 1. *For every positive integer* $m\geq 2$ *the cubic polynomial*

$$r_m(y)=-y^3+\frac{4(m^2+m+2)}{(m+2)^2}y^2-\frac{4(m^2-m+1)}{(m+2)^2}y+\frac{(m-1)^2}{(m+2)^2}$$

has three positive zeros. Moreover, the largest zero $y(m)$ *of* $r_m(y)$ *does not exceed* $y(\infty):=(3+\sqrt{5})/2$ *and* $y(m)\to y(\infty)$ *as* m *diverges.*

THEOREM 6. *Let* $n=2m+2$, $m\geq 2$ *be an even positive integer and, let* $y(m)$ *be defined as in Lemma 1. Then for every* $S_n(\theta)\in\mathcal{S}_n$

$$|b_{n-2}|\leq\sqrt{y(m)}.$$

In particular,

$$|b_{n-2}|<((3+\sqrt{5})/2)^{1/2}.$$

The method allows us to obtain also some non-negative cosine polynomials.

COROLLARY 1. *The following cosine polynomials are non-negative:*

$$1+2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{2k(n+k+1)}{n(n+2)}\right)\cos k\theta, \tag{2.10}$$

$$\begin{aligned}1+\cos\theta &+ 2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{k}{n+2}\right)\left(1+\frac{k}{2n+3}\right)\cos(2k)\theta\\ &+ \sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1+k\left(\frac{4(k+1)}{2n+3}-\frac{2k+1}{n+2}\right)\right)\cos(2k+1)\theta,\end{aligned} \tag{2.11}$$

$$\begin{aligned}1-\cos\theta &+ 2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{k}{n+2}\right)\left(1+\frac{k}{2n+3}\right)\cos(2k)\theta\\ &- \sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1+k\left(\frac{4(k+1)}{2n+3}-\frac{2k+1}{n+2}\right)\right)\cos(2k+1)\theta,\end{aligned} \tag{2.12}$$

$$1+2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{k}{n+2}\right)\left(1+\frac{k}{2n+3}\right)\cos k\theta, \tag{2.13}$$

$$1+2\sum_{k=0}^{n-1}\left(\left(1-\frac{k+1/2}{n+1/2}\right)\cos(2k+1)\theta+\left(1-\frac{k+1}{n+1/2}\right)\cos(2k+2)\theta\right), \tag{2.14}$$

$$1+2\sum_{k=0}^{n-1}\left(\left(-1+\frac{k+1/2}{n+1/2}\right)\cos(2k+1)\theta+\left(1-\frac{k+1}{n+1/2}\right)\cos(2k+2)\theta\right), \tag{2.15}$$

$$1+2\sum_{k=1}^{n-1}\left(1-\frac{k+1}{n+1/2}\right)\cos(k+1)\theta, \tag{2.16}$$

$$1+\frac{4n+2}{n+1}pq\cos\theta+2\sum_{k=1}^{n}\left\{\left(1-\frac{k}{n+1}\right)\cos 2k\theta+\left(1-\frac{2k-n}{n+1}\right)pq\cos(2k+1)\theta\right\}, \tag{2.17}$$

$$\begin{aligned}1 &+ \frac{4n-2}{n+1}pq\cos\theta+2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{2k(n+k+1)}{n(n+2)}\right)\cos 2k\theta\\ &+ 2\sum_{k=1}^{n}\frac{2n^3+3n^2-2n-6kn^2-12kn+4k^3+6k^2+2k}{n(n+1)(n+2)}pq\cos(2k+1)\theta,\end{aligned} \tag{2.18}$$

where the parameters p and q in (2.17) and (2.18) satisfy $p^2+q^2=1$,

$$\frac{3+\sqrt{5}}{4}(m-1)\left(1+2\sum_{k=1}^{m-2}\left(1-\frac{k}{m-1}\right)\cos k\theta\right)+1+(1+\sqrt{5})\sum_{k=1}^{m-2}\cos k\theta+\cos m\theta \tag{2.19}$$

and

$$
\begin{aligned}
& \frac{m(-5+2m+\sqrt{m(5m-8)})}{6\sqrt{m(5m-8)}} \\
+ & \sum_{k=1}^{m-1}\left(\frac{10k+2k^3-4m-3km^2+m^3}{3m^2-12}\right)\cos k\theta \\
+ & \sum_{k=1}^{m-1}\left(\frac{2m^4-6km^3+m^3-8m^2+4k^3m+20km-4m-4k^3+4k}{(3m^2-12)\sqrt{m(5m-8)}}\right)\cos k\theta \\
- & \left(\frac{m+\frac{(m-4)m}{\sqrt{m(5m-8)}}}{2(m+2)}\right)\cos m\theta.
\end{aligned}
\tag{2.20}
$$

The polynomial (2.13) is a convex combination of (2.11) and (2.12), and the polynomial (2.16) is a convex combination of (2.14) and (2.15).

The explicit forms of the non-negative cosine polynomials given by the formulae from (2.10) to (2.16) are somehow very similar to the classical Fejér kernel (1.1). While the coefficients of $K_n(\theta)$ approximate "linearly" the coefficients of the formal Fourier series of Dirac's delta, the coefficients of our cosine polynomials, except for (2.16), provide "quadratic", or even "cubic", approximations to the coefficients of (1.2). This intuitive observation suggests that the above cosine polynomials may be better approximations to the Dirac's delta. It turns out to be true. The graph of each of the polynomials, from (2.10) to (2.15) is much "closer" to $\delta(x)$ than the graph of (1.1). Even the polynomial (2.16), whose explicit form is very similar to the one of $K_n(\theta)$, approximates the Dirac's delta better than $K_n(\theta)$. Recent investigations show that these cosine polynomials are summability kernels which immediately implies convergence in $L_p[-\pi,\pi]$ of their convolutions with any function from the $L_p[-\pi,\pi]$-space.

Employing a result of Turán [**32**, Theorem I] we obtain another consequence of Theorem 3, Theorem 4 and Theorem 5 which provides interesting analogues of Fejér-Jackson-Gronwall's inequality.

Corollary 2. *The sine polynomials*

$$\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{k-1}{n(n+1)}\right)\sin k\theta, \tag{2.21}$$

$$\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{2(k-1)}{n^2-1}\right)\sin k\theta, \tag{2.22}$$

$$\sum_{k=1}^{n}\frac{1}{k}\sin k\theta \;-\; \frac{1}{2n}\sin n\theta,$$

$$\sin\theta+\left(\frac{5+2\sqrt{5}}{5}\right)\sum_{k=2}^{n-2}\frac{1}{k}\sin k\theta+\left(\frac{1+\sqrt{5}}{2(n-1)}\right)\sin(n-1)\theta+\left(\frac{5+\sqrt{5}}{10n}\right)\sin n\theta$$

and

$$
\begin{aligned}
\sin\theta \quad &+ \quad \sum_{k=2}^{n-2} \left(\frac{n^2-2n-2k^2+2k-3+\frac{2(n^2-1)(n-3)-4k(n-2)(k-1)}{\sqrt{(n-1)(5n-13)}}}{(n-3)(n+1)k} \right) \sin k\theta \\
&+ \quad \left(\frac{5-n-\sqrt{(n-1)(5n-13)}}{2(n+1)(n-1)} \right) \sin(n-1)\theta \\
&+ \quad \left(\frac{1-n-\frac{(n-5)(n-1)}{\sqrt{(n-1)(5n-13)}}}{2n(n+1)} \right) \sin n\theta
\end{aligned}
$$

are non-negative.

The graphs of the polynomials (2.21) and (2.22) show that their Gibb's phenomenon is almost neglectable in comparison with the Gibb's phenomenon of Fejér-Jackson-Gronwall's polynomials. Moreover, (2.21) and (2.22) approximate the function $(\pi-\theta)/2$ uniformly on every compact subset of $(0,\pi]$ much better than (1.3) does.

3. *Fejér-Riesz type representation of non-negative sine polynomials*

A careful inspection of the proof of Fejér-Riesz's theorem shows that the coefficients of any non-negative cosine polynomial are representable in terms of real parameters c_k.

LEMMA 2. *Let*

$$C_n(\theta) = a_0/2 + \sum_{k=1}^{n} a_k \cos k\theta$$

be a cosine polynomial of order n which is non-negative for every real θ. Then there exists an algebraic polynomial with real coefficients $R(z) = \sum_{k=0}^{n} c_k z^k$ of degree n such that $T(\theta) = |R(e^{i\theta})|^2$. Thus, the cosine polynomial $C_n(\theta)$ of order n is non-negative if and only if there exist real numbers c_k, $k = 0, 1, \ldots, n$, such that

$$
\begin{aligned}
a_0 &= \sum_{k=0}^{n} c_k^2, \\
a_k &= \sum_{\nu=0}^{n-k} c_{k+\nu} c_\nu \quad \text{for} \quad k = 1, \ldots, n.
\end{aligned}
\tag{3.1}
$$

The following relation between non-negative sine and non-negative cosine polynomials is a simple observation (see [**28**]).

LEMMA 3. *The sine polynomial of order n*

$$S_n(\theta) = \sum_{k=1}^{n} b_k \sin k\theta$$

is non-negative in $[0,\pi]$ if and only if and only if the non-negative cosine polynomial *of order* $n-1$

$$C_n(\theta) = a_0/2 + \sum_{k=1}^{n-1} a_k \cos k\theta,$$

where

$$\begin{aligned} b_k &= a_{k-1} - a_{k+1} \quad \text{for} \quad k = 1, \ldots, n-2, \\ b_{n-1} &= a_{n-2}, \\ b_n &= a_{n-1}, \end{aligned} \tag{3.2}$$

is nonnegative.

As a consequence of the last two lemmas we obtain a parametric representation for the coefficients of the non-negative sine polynomials.

Lemma 4. *The sine polynomial of order* n

$$S_n(\theta) = \sum_{k=1}^{n} b_k \sin k\theta$$

is non-negative if and only if there exist real numbers $c_0, \ldots, c_{n-1}$*, such that*

$$\begin{aligned} b_1 &= \sum_{\nu=0}^{n-1} c_\nu^2 - \sum_{\nu=0}^{n-3} c_\nu c_{\nu+2}, \\ b_k &= \sum_{\nu=0}^{n-k} c_{k+\nu-1} c_\nu - \sum_{\nu=0}^{n-k-2} c_{k+\nu+1} c_\nu, \quad \text{for} \quad k = 2, \ldots, n-2, \\ b_{n-1} &= c_0 c_{n-2} + c_1 c_{n-1}, \\ b_n &= c_0 c_{n-1}. \end{aligned} \tag{3.3}$$

The principal idea is to canonise the quadratic form, which represents b_1 in order to obtain sum of squares. The process of canonisation begins as follows:

$$\begin{aligned} b_1 &= \sum_{k=0}^{n-1} c_k^2 - \sum_{k=0}^{n-3} c_k c_{k+2} \\ &= (c_0 - \frac{1}{2}c_2)^2 + (c_1 - \frac{1}{2}c_3)^2 + \frac{3}{4}c_2^2 + \frac{3}{4}c_3^2 + \sum_{k=4}^{n-1} c_k^2 - \sum_{k=2}^{n-3} c_k c_{k+2} \\ &= (c_0 - \frac{1}{2}c_2)^2 + (c_1 - \frac{1}{2}c_3)^2 + (\frac{\sqrt{3}}{2}c_2 - \frac{1}{\sqrt{3}}c_4)^2 + (\frac{\sqrt{3}}{2}c_3 - \frac{1}{\sqrt{3}}c_5)^2 \\ &+ \frac{2}{3}c_4^2 + \frac{2}{3}c_5^2 + \sum_{k=6}^{n-1} c_k^2 - \sum_{k=4}^{n-3} c_k c_{k+2}. \end{aligned}$$

It is not difficult to see that if n is odd, $n = 2m+1$, then b_1 can be written as a sum of squares in the form

$$\begin{aligned} b_1 &= \sum_{k=0}^{m-2} \left\{ \left(\sqrt{q_{k+1}} c_{2k} - \frac{c_{2k+2}}{2\sqrt{q_{k+1}}} \right)^2 + \left(\sqrt{q_{k+1}} c_{2k+1} - \frac{c_{2k+3}}{2\sqrt{q_{k+1}}} \right)^2 \right\} \\ &+ \left(\sqrt{q_m} c_{2m-2} - \frac{c_{2m}}{2\sqrt{q_m}} \right)^2 + q_m c_{2m-1}^2 + q_{m+1} c_{2m}^2, \end{aligned}$$

and if n is even, $n = 2m+2$, then b_1 is given by

$$\begin{aligned} b_1 &= \sum_{k=0}^{m-1} \left\{ \left(\sqrt{q_{k+1}}c_{2k} - \frac{c_{2k+2}}{2\sqrt{q_{k+1}}} \right)^2 + \left(\sqrt{q_{k+1}}c_{2k+1} - \frac{c_{2k+3}}{2\sqrt{q_{k+1}}} \right)^2 \right\} \\ &+ \; q_{m+1}c_{2m}^2 + q_{m+1}c_{2m+1}^2, \end{aligned}$$

where the parameters $q_k,\ k = 1,\dots,m+1$, are determined by the recurrence relation $q_1 = 1,\ q_{k+1} = 1 - 1/(4q_k)$. The solution of this recurrence equation is

$$q_k = \frac{k+1}{2k}. \tag{3.4}$$

We can formulate our next technical lemma.

LEMMA 5. *Let the sequence $\{q_k\}$ be defined by (3.4). Then, for every positive integer n, the coefficient b_1 can be represented as a sum of squares*

$$b_1 = \sum_{k=0}^{n-1} d_k^2,$$

where, for n odd, $n = 2m+1$, the parameters d_k and c_k are related by

$$\begin{aligned} \sqrt{q_k}c_{2k-2} - \frac{1}{2}\frac{1}{\sqrt{q_k}}c_{2k} &= d_{2k-2}, \quad k = 1,\dots,m, \\ \sqrt{q_k}c_{2k-1} - \frac{1}{2}\frac{1}{\sqrt{q_k}}c_{2k+1} &= d_{2k-1}, \quad k = 1,\dots,m-1, \\ \sqrt{q_{m+1}}c_{2m} &= d_{2m}, \\ \sqrt{q_m}c_{2m-1} &= d_{2m-1}, \end{aligned}$$

and for n even, $n = 2m+2$, d_k and c_k are related by

$$\begin{aligned} \sqrt{q_k}c_{2k-2} - \frac{1}{2}\frac{1}{\sqrt{q_k}}c_{2k} &= d_{2k-2}, \quad k = 1,\dots,m, \\ \sqrt{q_k}c_{2k-1} - \frac{1}{2}\frac{1}{\sqrt{q_k}}c_{2k+1} &= d_{2k-1}, \quad k = 1,\dots,m, \\ \sqrt{q_{m+1}}c_{2m} &= d_{2m}, \\ \sqrt{q_{m+1}}c_{2m+1} &= d_{2m+1}. \end{aligned}$$

Solve the above systems of linear equations with respect to c_k. Having in mind the explicit form (3.4) of $q_k,\ k = 1,\dots$, and setting $q_{k+1}\cdots q_{k+\nu} = 1$ for $\nu = 0$, we obtain an explicit representation of c_k in terms of a linear combination of the new parameters $d_j,\ j = k, k+2, k+4, \dots$. Thus $c_{2k},\ k = 0,1,\dots,m$ are given by

$$\begin{aligned} c_{2k} &= \sum_{\nu=0}^{m-k} \frac{1}{2^\nu}(q_{k+1}\cdots q_{k+\nu}\sqrt{q_{k+\nu+1}})^{-1} d_{2k+2\nu} \\ &= (k+1)\sum_{\nu=0}^{m-k} \sqrt{\frac{2}{(k+\nu+1)(k+\nu+2)}}\, d_{2k+2\nu}, \end{aligned} \tag{3.5}$$

while for the parameters c_{2k+1} we have

$$\begin{aligned} c_{2k+1} &= \sum_{\nu=0}^{m-k-1} \frac{1}{2^\nu}(q_{k+1}\cdots q_{k+\nu}\sqrt{q_{k+\nu+1}})^{-1} d_{2k+2\nu+1} \\ &= (k+1)\sum_{\nu=0}^{m-k-1} \sqrt{\frac{2}{(k+\nu+1)(k+\nu+2)}} d_{2k+2\nu+1}, \end{aligned} \tag{3.6}$$

if $n = 2m+1$, or

$$\begin{aligned} c_{2k+1} &= \sum_{\nu=0}^{m-k} \frac{1}{2^\nu}(q_{k+1}\cdots q_{k+\nu}\sqrt{q_{k+\nu+1}})^{-1} d_{2k+2\nu+1} \\ &= (k+1)\sum_{\nu=0}^{m-k} \sqrt{\frac{2}{(k+\nu+1)(k+\nu+2)}} d_{2k+2\nu+1}, \end{aligned} \tag{3.7}$$

if $n = 2m+2$.

Lemma 4, Lemma 5 and relations (3.5), (3.6) and (3.7) yield the desired parametric representation of the coefficients of non-negative sine polynomials.

THEOREM 7. *The sine polynomial*

$$S_n(\theta) = \sum_{k=1}^{n} b_k \sin k\theta$$

is non-negative for $\theta \in [0,\pi]$ if and only if there exist real parameters $d_0,\dots,d_{n-1}$, such that $b_1 = d_0^2 + \cdots + d_{n-1}^2$ and

$$b_k = \sum_{i=0}^{n-1} \delta_{i,i}^{(k)} d_i^2 + 2 \sum_{0\le i<j\le n-1} \delta_{i,j}^{(k)} d_i d_j \quad \text{for} \quad k = 1,\dots,n-1, \tag{3.8}$$

where the coefficients $\delta_{i,j}^{(k)}$ are explicitly obtained by the relations (3.3), (3.5), (3.6) *and* (3.7).

Let $D_k = (\delta_{i,j}^{(k)})_{i,j=0}^{n-1}$ be symmetric matrix associated with the quadratic form (3.8). Denote by $\mathbf{d}$ the vector $(d_0,\dots,d_{n-1})^T$. Then

$$b_k = \mathbf{d}^T D_k \mathbf{d}.$$

By the Rayleigh-Ritz theorem (see Theorem 4.2.2 on p. 176 in Horn and Johnson [**19**]) we obtain the main result in this section.

THEOREM 8. *Let $\lambda_{k,min}$ and $\lambda_{k,max}$ be the smallest and the largest eigenvalues of D_k. If $S_n(\theta) \in \mathcal{S}_n$, then*

$$\lambda_{k,min} \le b_k \le \lambda_{k,max}.$$

Moreover, the equality $b_k = \lambda_{k,max}$ is attained if and only if $S_n(\theta)$ is a sine polynomial in $\mathcal{S}_n$ whose coefficients b_i, $i = 2,\dots,n$ are obtained by the formulae

$$b_i = \mathbf{d}_{k,max}^T D_i \mathbf{d}_{k,max},$$

where $\mathbf{d}_{k,max}$ is an eigenvector with unit length, associated with the eigenvalue $\lambda_{k,max}$.

Similarly, the equality $b_k = \lambda_{k,min}$ *is attained if and only if* $S_n(\theta)$ *is a sine polynomial in* $\mathcal{S}_n$*, whose coefficients* $b_i,\ i = 2, \ldots, n$ *are obtained by the formulae*

$$b_i = \mathbf{d}_{k,min}^T D_i \mathbf{d}_{k,min},$$

where $\mathbf{d}_{k,min}$ *is an eigenvector with unit length, associated with the eigenvalue* $\lambda_{k,min}$.

4. *Extremal polynomials for* b_n *and* b_{n-1}.

In this section we find the extremal values and the corresponding extremal polynomials for the coefficients b_n and b_{n-1}. It is worth mentioning that the extremal values as well as the associated extremal polynomials depend on the parity of n. Thus, for any coefficient we consider four cases: the problems of determining the minimum and the maximum and, for each of these, the cases of even and odd n.

4.1 *The minimum and the maximum of* b_n *for* $n = 2m+1$ *and the extremal polynomials associated with* $b_{n,min}$ *and* $b_{n,max}$.

On using the relations (3.3), (3.5) and (3.6) we obtain

$$b_n = \sum_{k=0}^{m} \frac{d_{2k} d_{2m}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}} \sqrt{q_{m+1}}},$$

where q_k is given by (3.4). Since

$$2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}} \sqrt{q_{m+1}} = \sqrt{\frac{m+2}{m+1}} \frac{\sqrt{(k+1)(k+2)}}{2},$$

then

$$b_n = \sqrt{\frac{m+1}{m+2}} \sum_{k=0}^{m} \frac{2 d_{2k} d_{2m}}{\sqrt{(k+1)(k+2)}}.$$

Set

$$\alpha := \sqrt{\frac{m+1}{m+2}} \tag{4.1}$$

and

$$A_j := \frac{1}{\sqrt{(j+1)(j+2)}}, \quad j = 0, \ldots, m. \tag{4.2}$$

Then the symmetric matrix $D_n^{(o)}$ associated with the quadratic form which represents b_n is

$$D_n^{(o)} = \begin{pmatrix} 2\alpha A_m & 0 & \alpha A_{m-1} & \cdots & \alpha A_1 & 0 & \alpha A_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha A_{m-1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \\ \alpha A_1 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ \alpha A_0 & 0 & 0 & & & & \end{pmatrix}.$$

Hence the characteristic polynomial of $D_n^{(o)}$ is

$$\begin{aligned} det(D_n^{(o)} - \lambda I) &= (-\lambda)^{2m} \left\{(\alpha^2 A_0^2 + \cdots + \alpha^2 A_{m-1}^2)/\lambda + 2\alpha A_m - \lambda\right\} \\ &= (-\lambda)^{2m-1} \left\{\lambda^2 - 2\frac{\alpha}{\sqrt{(m+1)(m+2)}}\lambda - \alpha^2 \frac{m}{m+1}\right\}. \end{aligned}$$

Since the smallest and the largest zeros of this polynomial are $-m/(m+2)$ and 1, then by Theorem 8

$$-\frac{m}{m+2} \le b_n \le 1, \quad m = (n-1)/2.$$

In order to determine the extremal polynomial associated with $b_{n,max}$ we need to find first an eigenvector $\mathbf{d}_{n,max}^{(o)}$ of $D_n^{(o)}$, associated with the eigenvalue $\lambda = 1$, such that

$$d_0^2 + \cdots + d_{2m}^2 = 1. \tag{4.3}$$

From the homogeneous linear system

$$\begin{pmatrix} 2\alpha A_m - 1 & 0 & \alpha A_{m-1} & \cdots & \alpha A_1 & 0 & \alpha A_0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \alpha A_{m-1} & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \vdots \\ \alpha A_1 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ \alpha A_0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} d_{2m} \\ d_{2m-1} \\ d_{2m-2} \\ \vdots \\ d_2 \\ d_1 \\ d_0 \end{pmatrix} = 0$$

for the coordinates of $\mathbf{d}_{n,max}^{(o)}$, we obtain $d_{2k} = \alpha A_k d_{2m}$ and $d_{2k+1} = 0$ for $k = 0, \ldots, m-1$. A substitution of these values of d_j in (4.3) yields $d_{2m} = \pm((m+2)/(2m+2))^{1/2}$. Thus the desired eigenvectors are $\pm\mathbf{d}_{n,max}^{(o)}$, where

$$\mathbf{d}_{n,max}^{(o)} = \frac{1}{\sqrt{2}}\left(A_0, 0, A_1, 0, \ldots, A_{m-2}, 0, A_{m-1}, 0, 1/\alpha\right).$$

Observe that both the coefficients a_k and b_k are expressed as quadratic forms of $d_0, d_1, \ldots, d_{2m}$. Thus any choice of the sign of the eigenvector will imply the same extremal polynomial. We prefer working with $\mathbf{d}_{n,max}^{(o)}$. From (3.5) and (3.6) we obtain $c_{2k+1} = 0$ for $k = 0, \ldots, m-1$, and

$$\begin{aligned} c_{2k} &= (k+1)\sum_{\nu=0}^{m-k} \sqrt{\frac{2}{(k+\nu+1)(k+\nu+2)}}\, d_{2k+2\nu} \\ &= \frac{k+1}{m+1} + (k+1)\sum_{\nu=0}^{m-k-1} \frac{1}{(k+\nu+1)(k+\nu+2)} \\ &= 1. \end{aligned}$$

Then the expressions (3.1) yield $a_{2k} = m+1-k$ for $k = 0, \ldots, m$ and $a_{2k+1} = 0$ for $k = 0, \ldots, m-1$. Thus we obtained the classical non-negative cosine polynomial (1.1). On using (3.2) we obtain (1.5).

The procedure of determining extremal sine polynomial associated with $b_{n,min}$ is similar. First we find the eigenvectors of $D_n^{(o)}$, associated with the smallest eigenvalue $\lambda = -m/(m+2)$. They are $\pm\mathbf{d}_{n,min}$, where

$$\mathbf{d}_{n,min} = \sqrt{\frac{m+2}{2m}}\,(A_0, 0, A_1, 0, \ldots, 0, A_{m-1}, 0, -mA_m)^T\,,$$

and we choose the positive sign. Thus, $d_{2k+1} = 0$, $d_{2m} = -\sqrt{m/(2m+2)}$, and

$$d_{2k+2\nu} = \sqrt{\frac{m+2}{2m}}\frac{1}{(k+\nu+1)(k+\nu+2)} \quad \text{for} \quad k+\nu = 0, \ldots, m-1.$$

Then, by (3.6) we obtain $c_{2k+1} = 0$, and by (3.5) we get

$$\begin{aligned} c_{2k} &= (k+1)\sum_{\nu=0}^{m-k}\sqrt{\frac{2}{(k+\nu+1)(k+\nu+2)}}d_{2k+2\nu} \\ &= -\frac{k+1}{m+1}\sqrt{\frac{m}{m+2}} + (k+1)\sqrt{\frac{m}{m+2}}\sum_{\nu=0}^{m-k-1}\frac{1}{(k+\nu+1)(k+\nu+2)} \\ &= \sqrt{\frac{m}{m+2}}\frac{m-2k}{m} \quad \text{for} \quad k = 0, \ldots, m. \end{aligned}$$

Hence $a_{2k+1} = 0$ and the general formula for a_{2k} in terms of $c_0, c_2, \ldots, c_{2m}$ yields

$$\begin{aligned} a_{2k} &= \frac{1}{m(m+2)}\sum_{j=0}^{m-k}(m-2j)(m-2j-2k) \\ &= \frac{(m-k+1)(m^2+2m-2mk-2k^2-2k)}{3m(m+2)}. \end{aligned}$$

We arrived at the non-negative cosine polynomial (2.10). Now we need to express the coefficients b_k in terms of a_k by (3.2). Obviously $b_{2k} = 0$ and

$$b_{2k+1} = 1 - \frac{2k(k+1)}{m(m+2)}.$$

This gives the non-negative sine polynomial (2.5).

4.2 *The minimum and the maximum of b_n for $n = 2m+2$ and the extremal polynomials associated with $b_{n,min}$ and $b_{n,max}$.*

Relations (3.3), (3.5) and (3.7) imply

$$b_n = \sum_{k=0}^{m}\frac{d_{2k}d_{2m+1}}{2^k q_1 q_2 \cdots q_k\sqrt{q_{k+1}}\sqrt{q_{m+1}}}.$$

Since

$$2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_{m+1}} = \sqrt{\frac{m+2}{m+1}}\frac{\sqrt{(k+1)(k+2)}}{2},$$

then

$$b_n = \sqrt{\frac{m+1}{m+2}}\sum_{k=0}^{m}\frac{2d_{2k}d_{2m+1}}{\sqrt{(k+1)(k+2)}}.$$

Hence the symmetric matrix $D_n^{(e)}$ associated with the quadratic form which represents b_n is

$$D_n^{(e)} = \begin{pmatrix} 0 & \alpha A_m & 0 & \cdots & \alpha A_1 & 0 & \alpha A_0 \\ \alpha A_m & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \\ \alpha A_1 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ \alpha A_0 & 0 & 0 & & & & \end{pmatrix},$$

where α and A_j are defined by (4.1) and (4.2). The characteristic polynomial of $D_n^{(e)}$ is

$$\begin{aligned} det(D_n^{(e)} - \lambda I) &= (-\lambda)^{2m+1}\left\{(\alpha^2 A_0^2 + \cdots + \alpha^2 A_{m-1}^2 + \alpha^2 A_m^2)/\lambda - \lambda\right\} \\ &= (-\lambda)^{2m}\left\{\lambda^2 - \alpha^2\frac{m+1}{m+2}\right\}. \end{aligned}$$

Since the smallest and the largest zeros of this polynomial are $-(m+1)/(m+2)$ and $(m+1)/(m+2)$, then

$$-\frac{m+1}{m+2} \le b_n \le \frac{m+1}{m+2}, \quad m = (n-2)/2.$$

First we obtain an eigenvector $\mathbf{d}_{n,max}^{(e)}$ of $D_n^{(e)}$, associated with the eigenvalue $\lambda = (m+1)/(m+2)$, such that

$$d_0^2 + \cdots + d_{2m+1}^2 = 1. \tag{4.4}$$

Solve the homogeneous linear system $(D_n^{(e)} - ((m+1)/(m+2))I)\mathbf{d}_{n,max}^{(e)} = 0$. We obtain $d_{2k} = ((m+2)/(m+1))\alpha A_k d_{2m+1}$, $k = 0,\ldots,m$ and $d_{2k+1} = 0$, $k = 0,\ldots,m-1$. Substitute these values of d_j in (4.4) to get $d_{2m+1} = \pm\sqrt{2}/2$. Thus the eigenvectors we need are $\pm\mathbf{d}_{n,max}^{(e)}$, where

$$\mathbf{d}_{n,max}^{(e)} = \frac{1}{\sqrt{2}\alpha}\left(A_0, 0, A_1, 0, \ldots, A_m, \alpha\right)^T.$$

We shall work with $\mathbf{d}_{n,max}^{(e)}$. By (3.5) and (3.7) we obtain

$$c_{2k} = \frac{m-k+1}{m+2}\sqrt{\frac{m+2}{m+1}} \quad \text{for} \quad k = 0,\ldots,m,$$

and

$$c_{2k+1} = \frac{k+1}{m+1}\sqrt{\frac{m+1}{m+2}} \quad \text{for} \quad k = 0,\ldots,m.$$

Then formulae (3.1) imply that

$$a_{2k} = \frac{(m-k+1)(2m^2 - mk + 7m - k^2 - k + 6)}{3(m+1)(m+2)}$$

and

$$a_{2k+1} = \frac{(2m^3 + 9m^2 + 13m - 2k^3 - 3k^2 - k + 6)}{6(m+1)(m+2)}.$$

Thus we obtained the non-negative cosine polynomial (2.11).On using relations (3.2) we obtain the non-negative sine polynomial (2.1).

In order to determine the extremal sine polynomial associated with $b_{n,min}$, we find the eigenvectors of $D_n^{(e)}$, associated with the smallest eigenvalue $\lambda = -(m+1)/(m+2)$. They are $\pm \mathbf{d}_{n,min}^{(e)}$, where

$$\mathbf{d}_{n,min}^{(e)} = -\frac{1}{\sqrt{2}\alpha}\left(A_0, 0, A_1, 0, \ldots, A_m, -\alpha\right)^T,$$

and we choose the positive sign. Then formulae (3.5) and (3.7) yield

$$c_{2k} = -\frac{m-k+1}{m+2}\sqrt{\frac{m+2}{m+1}} \quad \text{and} \quad c_{2k+1} = \frac{k+1}{m+1}\sqrt{\frac{m+1}{m+2}}$$

for $k = 0, \ldots, m$. Then,

$$a_{2k} = \frac{(m-k+1)(2m^2 - mk + 7m - k^2 - k + 6)}{3(m+1)(m+2)}$$

and

$$a_{2k+1} = -\frac{(2m^3 + 9m^2 + 13m - 2k^3 - 3k^2 - k + 6)}{6(m+1)(m+2)}.$$

Thus we obtain the non-negative cosine polynomial (2.12).On using the relations (3.2) we obtain the non-negative sine polynomial (2.2).

4.3 *The minimum and the maximum of b_{n-1} for $n = 2m+1$ and the extremal polynomials associated with $b_{n,min}$ and $b_{n,max}$.*

In this case

$$\begin{aligned} b_{n-1} &= \sum_{k=0}^{m} \frac{d_{2k} d_{2m-1}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_m}} + \sum_{k=0}^{m-1} \frac{d_{2k+1} d_{2m}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_{m+1}}} \\ &= \sqrt{\frac{m}{m+1}} \sum_{k=0}^{m} \frac{2 d_{2k} d_{2m-1}}{\sqrt{(k+1)(k+2)}} + \sqrt{\frac{m+1}{m+2}} \sum_{k=0}^{m-1} \frac{2 d_{2k+1} d_{2m}}{\sqrt{(k+1)(k+2)}}. \end{aligned}$$

Setting

$$\beta := \sqrt{\frac{m}{m+1}}, \tag{4.5}$$

we obtain the symmetric matrix

$$D_{n-1}^{(o)} = \begin{pmatrix} 0 & \beta A_m + \alpha A_{m-1} & 0 & \cdots & \alpha A_0 & 0 \\ \beta A_m + \alpha A_{m-1} & 0 & \beta A_{m-1} & \cdots & 0 & \beta A_0 \\ 0 & \beta A_{m-1} & & & & \\ \alpha A_{m-2} & 0 & & & & \\ \vdots & \vdots & & & & \\ 0 & \beta A_1 & & & & \\ \alpha A_0 & 0 & & & & \\ 0 & \beta A_0 & & & & \end{pmatrix}$$

associated with the quadratic form which represents b_{n-1}. Hence the characteristic polynomial of $D^{(o)}_{n-1}$ is

$$det(D^{(o)}_{n-1} - \lambda I) = (-\lambda)^{2m-3}\left\{\lambda^4 - \frac{2m^2+2m+2}{(m+1)(m+2)}\lambda^2 + \frac{m(m-1)}{(m+1)(m+2)}\right\},$$

whose smallest and largest zeros are ± 1. By Theorem 8 we have

$$|b_{n-1}| \le 1, \quad m = (n-1)/2.$$

In order to determine the extremal polynomial associated with $b_{n-1,max}$ we need to find an eigenvector $\mathbf{d}^{(o)}_{n-1,max}$ of $D^{(o)}_{n-1}$, associated with the eigenvalue $\lambda = 1$, such that $d_0^2 + \cdots + d_{2m}^2 = 1$. For the eigenvector $\mathbf{d}^{(o)}_{n-1,max}$, we obtain

$$\mathbf{d}^{(o)}_{n-1,max} = \frac{1}{2}\left(A_0, A_0, \ldots, A_{m-2}, A_{m-2}, A_{m-1}, \sqrt{\frac{m+1}{m}}, \sqrt{\frac{m+2}{m+1}}\right)^T.$$

The relations (3.5) and (3.6) yield $c_k = \sqrt{2}/2$ for $k = 0, \ldots, 2m$. Then the expressions (3.1) imply $a_{2k} = m - k + 1/2$ for $k = 0, \ldots, m$ and $a_{2k+1} = m - k$ for $k = 0, \ldots, m-1$. We obtain the non-negative cosine polynomial (2.14) and, by (3.2), the positive sine polynomial (2.6).

The extremal sine polynomial associated with $b_{n-1,min}$ is obtained trough the eigenvectors of $D^{(o)}_{n-1}$, associated with the smallest eigenvalue $\lambda = -1$. They are $\pm\mathbf{d}^{(e)}_{n,min}$, where

$$\mathbf{d}^{(o)}_{n-1,min} = \frac{1}{2}\left(A_0, -A_0, \ldots, A_{m-2}, -A_{m-2}, A_{m-1}, -\sqrt{\frac{m+1}{m}}, \sqrt{\frac{m+2}{m+1}}\right)^T.$$

The relations (3.5) and (3.6) yield $c_{2k} = \sqrt{2}/2$ for $k = 0, \ldots, m$ and $c_{2k+1} = -\sqrt{2}/2$ for $k = 0, \ldots, m-1$. Then the expressions (3.1) imply $a_{2k} = m - k + 1/2$ for $k = 0, \ldots, m$ and $a_{2k+1} = -(m-k)$ for $k = 0, \ldots, m-1$. Thus we obtain the non-negative cosine polynomial (2.15). Then, by (3.2), we obtain the non-negative sine polynomial (2.7).

4.4 *The minimum and the maximum of b_{n-1} for $n = 2m+2$ and the extremal polynomials associated with $b_{n-1,min}$ and $b_{n-1,max}$.*

Since in this case

$$\begin{aligned} b_{n-1} &= \sum_{k=0}^{m}\left(\frac{d_{2k}d_{2m}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_{m+1}}} + \frac{d_{2k+1}d_{2m+1}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_{m+1}}}\right) \\ &= \sqrt{\frac{m+1}{m+2}}\sum_{k=0}^{m}\frac{2(d_{2k}d_{2m} + d_{2k+1}d_{2m+1})}{\sqrt{(k+1)(k+2)}}, \end{aligned}$$

then the symmetric matrix $D^{(e)}_{n-1}$ associated with the quadratic form which represents b_{n-1} is

$$D^{(e)}_{n-1} = \begin{pmatrix} 2\alpha A_m & 0 & \alpha A_{m-1} & 0 & \cdots & \alpha A_0 & 0 \\ 0 & 2\alpha A_m & 0 & \alpha A_{m-1} & \cdots & 0 & \alpha A_0 \\ \alpha A_{m-1} & 0 & & & & & \\ 0 & \alpha A_{m-1} & & & & & \\ \vdots & \vdots & & & & & \\ \alpha A_0 & 0 & & & & & \\ 0 & \alpha A_0 & & & & & \end{pmatrix}.$$

Hence the characteristic polynomial of $D^{(e)}_{n-1}$ is

$$det(D^{(e)}_{n-1} - \lambda I) = \lambda^{2m-2}\left(\lambda^2 - \frac{2}{m+2}\lambda - \frac{m}{m+2}\right)^2,$$

whose smallest and largest zeros are $-m/(m+2)$ and 1. Therefore

$$-\frac{m}{m+2} \le b_{n-1} \le 1, \quad m = (n-2)/2.$$

In order to determine the extremal polynomial associated with $b_{n-1,max}$ we need to find first an eigenvector $\mathbf{d}^{(e)}_{n-1,max}$ of $D^{(e)}_{n-1}$, associated with the eigenvalue $\lambda = 1$, such that

$$d_0^2 + \cdots + d_{2m+1}^2 = 1.$$

As in the previous cases, for the coordinates of the eigenvector $\mathbf{d}^{(o)}_{n-1,max}$, we obtain $d_{2m} = P$, $d_{2m+1} = Q$, and $d_{2k} = \alpha P A_k$ and $d_{2k+1} = \alpha Q A_k$ for $k = 0, \dots, m-1$, where the parameters P and Q satisfy $P^2 + Q^2 = (m+2)/(2m+2)$. Now the relations (3.5) and (3.7) yield $c_{2k} = \sqrt{2}\alpha P$ and $c_{2k+1} = \sqrt{2}\alpha Q$ for $k = 0, \dots, m$. Then the expressions (3.1) imply

$$a_{2k} = m - k + 1 \quad \textit{and} \quad a_{2k+1} = \frac{2(m+1)(2m-2k+1)}{m+2}PQ \quad \text{for} \quad k = 0, \dots, m.$$

On substituting

$$P = \sqrt{\frac{m+2}{2m+2}}p \quad \text{and} \quad Q = \sqrt{\frac{m+2}{2m+2}}q$$

we obtain the non-negative cosine polynomial (2.17) and the relations (3.2) yield the non-negative sine polynomial (2.3).

The eigenvector $\mathbf{d}^{(e)}_{n-1,min}$ of $D^{(e)}_{n-1}$, associated with the smallest eigenvalue $\lambda = -m/(m+2)$ is given by $d_{2m} = P$, $d_{2m+1} = Q$, and $d_{2k} = -\alpha P A_k(m+2)/m$ and $d_{2k+1} = \alpha Q A_k(m+2)/m$ for $k = 0, \dots, m-1$, where in this case the parameters P and Q satisfy $P^2 + Q^2 = m/(2m+2)$. Now the relations (3.5) and (3.7) yield $c_{2k} = \sqrt{2}\alpha P(2k-m)/m$ and $c_{2k+1} = \sqrt{2}\alpha Q(2k-m)/m$ for $k = 0, \dots, m$. By (3.1) we obtain

$$a_{2k} = \frac{(m-k+1)(m^2 - 2km + 2m - 2k^2 - 2k)}{3m(m+2)} \quad \text{for} \quad k = 0, \dots, m,$$

and

$$a_{2k+1} = \frac{2(m+1)(4k^3 + 6k^2 + 2k - 2m - 12km + 3m^3 - 6km^2 + 2m^3)}{3m^2(m+2)}PQ$$

for $k = 0, \dots, m$. Substituting

$$P = \sqrt{\frac{m}{2m+2}}\,p \quad \text{and} \quad Q = \sqrt{\frac{m}{2m+2}}\,q$$

we obtain the non-negative cosine polynomial (2.18) and then, by (3.2), we obtain the non-negative sine polynomial (2.4).

5. *Extremal problems for b_{n-2}.*

5.1 *The minimum and the maximum of b_{n-2} for $n = 2m+1$ and the extremal polynomials associated with $b_{n-2,min}$ and $b_{n-2,max}$.*

Proof of Theorem 5. In this case (3.3), (3.5) and (3.6) imply

$$\begin{aligned} b_{n-2} &= \sum_{k=0}^{m} \left(\frac{d_{2k} d_{2m-2}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}} \sqrt{q_m}} + \frac{d_{2k} d_{2m}}{2^{k+1} q_1 q_2 \cdots q_k q_m \sqrt{q_{k+1}} \sqrt{q_{m+1}}} \right) \\ &+ \sum_{k=0}^{m} \frac{(-1)^{2^k} d_{2k} d_{2m}}{2^k q_1 q_2 \cdots q_k q_m \sqrt{q_{k+1}} \sqrt{q_{m+1}}} + \sum_{k=0}^{m-1} \frac{d_{2k+1} d_{2m-1}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}} \sqrt{q_m}} \\ &= \sum_{k=0}^{m} \frac{2}{\sqrt{(k+1)(k+2)}} \left(\beta d_{2k} d_{2m-2} + \gamma d_{2k} d_{2m} + (-1)^{2^k} \alpha d_{2k} d_{2m} \right) \\ &+ \sum_{k=0}^{m-1} \frac{2\beta d_{2k+1} d_{2m-1}}{\sqrt{(k+1)(k+2)}}, \end{aligned}$$

where α and β are defined by (4.1) and (4.5) and

$$\gamma := \frac{m}{\sqrt{(m+1)(m+2)}}. \tag{5.1}$$

Thus, the symmetric matrix $D^{(o)}_{n-2}$ associated with the quadratic form which represents b_{n-2} is

$$\begin{pmatrix} 2(\alpha+\gamma)\alpha A_m & 0 & (\alpha+\gamma)A_{m-1} + \alpha A_m & \cdots & (\gamma-\alpha)A_0 \\ 0 & 2\beta A_{m-1} & 0 & \cdots & 0 \\ (\alpha+\gamma)A_{m-1} + \alpha A_m & 0 & 2\beta A_{m-1} & \cdots & \beta A_0 \\ \vdots & \vdots & \vdots & & \\ (\alpha+\gamma)A_1 & 0 & \beta A_1 & & \\ 0 & \beta A_0 & 0 & & \\ (\gamma-\alpha)A_0 & 0 & \beta A_0 & & \end{pmatrix}.$$

Then, by straightforward calculations, we obtain the characteristic polynomial of $D^{(o)}_{n-2}$ in the form

$$-\lambda^{2m-5}(\lambda-1)\left(\lambda^2-\lambda-1\right)\left(\lambda+\frac{m-1}{m+2}\right)\left(\lambda^2+\frac{m-4}{m+2}\lambda-\frac{m-2}{m+2}\right).$$

It is easy to see that the smallest and the largest zeros of this polynomial are

$$\frac{-m+4-\sqrt{m(5m-8)}}{2(m+2)} \quad \text{and} \quad \frac{1+\sqrt{5}}{2}.$$

In order to determine the extremal polynomial associated with $b_{n-2,max}$ we to find an eigenvector $\mathbf{d}^{(o)}_{n-2,max}$ of D^{o}_{n-2}, associated with the eigenvalue $\lambda = (1+\sqrt{5})/2$, such that

$$d_0^2 + \cdots + d_{2m}^2 = 1. \tag{5.2}$$

Solve the homogeneous linear system $(D^{(o)}_{n-2} - ((1+\sqrt{5})/2)I)\mathbf{d}^{(o)}_{n-2,max} = 0$.
Set

$$\Psi := \sqrt{\frac{7+3\sqrt{5}}{5+2\sqrt{5}}}.$$

For the coordinates of $\mathbf{d}^{(o)}_{n-2,max}$ we obtain $d_{2k+1} = 0$ for $k = 0, \cdots, m-1$ and

$$\begin{aligned}
d_0 &= \frac{\Psi}{\sqrt{2}\lambda(1+\sqrt{5})}, \\
d_{2k} &= \frac{1+\sqrt{5}}{4} A_k \Psi \quad \text{for} \quad k = 1, \cdots, m-2, \\
d_{2m-2} &= \frac{2m+3+\sqrt{5}}{2\sqrt{m(m+1)}(1+\sqrt{5})}\Psi, \\
d_{2m} &= \frac{1}{2}\sqrt{\frac{m+2}{m+1}}\Psi.
\end{aligned}$$

A substitution of these values of d_j in (5.2) yields $d_{2m} = \pm(1/2)((m+2)/(m+1))^{1/2}\Psi$. Thus the desired eigenvectors are $\pm\mathbf{d}^{(o)}_{n-2,max}$, where

$$\mathbf{d}^{(o)}_{n-2,max} = \frac{\Psi}{2}\left(\frac{\sqrt{2}}{\lambda(1+\sqrt{5})}, 0, \cdots, 0, \frac{2m+3+\sqrt{5}}{(1+\sqrt{5})\sqrt{m(m+1)}}, 0, \sqrt{\frac{m+2}{m+1}}\right)^T.$$

We shall work with $\mathbf{d}^{(o)}_{n-2,max}$. Relations (3.5) and (3.6) yield $c_0 = c_{2m} = \Psi/\sqrt{2}$ and $c_{2k} = \sqrt{2}(1+\sqrt{5})\Psi/4$ for $k = 1, \cdots, m-1$. Then the relations (3.1) imply

$$\begin{aligned}
a_0 &= \Psi^2(1 + (\frac{3+\sqrt{5}}{4})(m-1), \\
a_{2k} &= \Psi^2((\frac{1+\sqrt{5}}{2}) + (\frac{3+\sqrt{5}}{4})(m-k-1)) \quad \text{for} \quad k = 1, \cdots, m-1, \\
a_{2m} &= \Psi^2/2 \quad \text{and} \\
a_{2k+1} &= 0 \quad \text{for} \quad k = 0, \cdots, m-1.
\end{aligned}$$

We obtain the non-negative cosine polynomial (2.19). On using (3.2) we obtain the non-negative sine polynomial (2.8).

In order to determine the extremal sine polynomial associated with $b^{(o)}_{n-2,min}$ we find the eigenvector $\mathbf{d}^{(o)}_{n-2,min}$ of $D^{(o)}_{n-2}$, associated with the smallest eigenvalue $\lambda = (-m+4-\sqrt{m(5m-8)})/(2(m+2))$.
Set

$$\Phi := \sqrt{\frac{m + \frac{m(m-4)}{\sqrt{m(5m-8)}}}{m+1}}.$$

For the coordinates of $\mathbf{d}^{(o)}_{n-2,min}$ we obtain $d_{2k+1}=0$ for $k=0,\cdots,m-1$ and

$$\begin{aligned}
d_0 &= -\frac{3m-\sqrt{m(5m-8)}}{4m\sqrt{\frac{2m+4}{m+1}}}\Phi,\\
d_{2k} &= \frac{(m+1)\sqrt{\frac{m+2}{m+1}}}{m-\sqrt{m(5m-8)}}\Phi A_k \quad \text{for} \quad k=1,\cdots,m-2,\\
d_{2m-2} &= \frac{m(1-m)+(1+m)\sqrt{m(5m-8)}}{4m\sqrt{m(m+2)}}\Phi,\\
d_{2m} &= \frac{\Phi}{2}.
\end{aligned}$$

By (3.5) and (3.6) we obtain $c_{2k+1}=0$ for $k=0,\cdots,m-1$ and

$$\begin{aligned}
c_0 &= -\sqrt{\frac{m+1}{2m+4}}\Phi,\\
c_{2k} &= \frac{(m-2k)(m+\sqrt{m(5m-8)})}{4m(2-m)\sqrt{\frac{m+2}{2m+2}}}\Phi \quad \text{for} \quad k=1,\cdots,m-1,\\
c_{2m} &= -c_0.
\end{aligned}$$

Then expressions (3.1) imply

$$\begin{aligned}
a_0 &= \frac{m(2m-5+\sqrt{m(5m-8)})}{3\sqrt{m(5m-8)}},\\
a_{2k} &= \frac{10k+2k^3-4m-3km^2+m^3}{3(m^2-4)}\\
&\quad+\frac{4k^3(m-1)+m(m^2-4)(2m+1)+k(4+20m-6m^3)}{3(m^2-4)\sqrt{m(5m-8)}},\\
a_{2m} &= -\frac{m+\frac{m(m-4)}{\sqrt{m(5m-8)}}}{4+2m}.
\end{aligned}$$

We obtain the non-negative cosine polynomial (2.20) and, by (3.2), the non-negative sine polynomial (2.9).

5.2 *The extrema of* b_{n-2} *for* $n=2m+2$.

Proof of Lemma 1. Simple analysis shows that $r_m(y)$ has two positive points of extrema and the values of $r_m(y)$ at these points are negative and positive, respectively. Since $r_m(0)>0$ and $r_m(y)$ is negative for sufficiently large y, then $r_m(y)$ has three positive zeros. On the other hand, $y(\infty)=(3+\sqrt{5})/2$ is strictly greater than the point of local maximum of r_m and $r_m(y(\infty))<0$. This implies the inequality $y(m)<y(\infty)$. It is clear that $r_m(y)$ converges uniformly on the compacts of the complex plane to the polynomial $-y^3+4y^2-4y+1$. Then by a theorem of Hurwitz (see Theorem 3.45 on p. 119 in [**31**]) the sequence of the largest zeros $y(m)$ of $r_m(y)$ converges to the largest zero $(3+\sqrt{5})/2$ of $-y^3+4y^2-4y+1$ as m diverges.

Proof of Theorem 6. By (3.3), (3.5) and (3.7) we obtain

$$\begin{aligned}
b_{n-2} &= \sum_{k=0}^{m}\left(\frac{d_{2k}d_{2m-1}}{2^k q_1 q_2\cdots q_k\sqrt{q_{k+1}}\sqrt{q_m}}+\frac{d_{2k}d_{2m+1}}{2^{k+1}q_1q_2\cdots q_k q_m\sqrt{q_{k+1}}\sqrt{q_{m+1}}}\right)\\
&+\sum_{k=0}^{m}\left(\frac{(-1)^{2^k}d_{2k}d_{2m+1}}{2^k q_1q_2\cdots q_k\sqrt{q_{k+1}}\sqrt{q_{m+1}}}+\frac{d_{2k+1}d_{2m}}{2^k q_1q_2\cdots q_k\sqrt{q_{k+1}}\sqrt{q_{m+1}}}\right)\\
&= 2\sum_{k=0}^{m}\frac{\beta d_{2k}d_{2m-1}+(\gamma+(-1)^{2^k}\alpha)d_{2k}d_{2m+1}+\alpha d_{2k+1}d_{2m}}{\sqrt{(k+1)(k+2)}},
\end{aligned}$$

where α, β and γ are defined by (4.1), (4.5) and (5.1). Thus, the symmetric matrix $D^{(e)}_{n-2}$ associated with the quadratic form which represents b_{n-2} is

$$\begin{pmatrix}
0 & e_1 & 0 & (\alpha+\gamma)A_{m-1} & \cdots & (\gamma-\alpha)A_0\\
e_1 & 0 & e_2 & 0 & \cdots & 0\\
0 & e_2 & 0 & \beta A_{m-1} & \cdots & \beta A_0\\
\vdots & \vdots & \vdots & & & \\
(\alpha+\gamma)A_1 & 0 & \beta A_1 & & & \\
0 & \alpha A_0 & 0 & & & \\
(\gamma-\alpha)A_0 & 0 & \beta A_0 & & &
\end{pmatrix},$$

where $e_1 = (2\alpha+\gamma)A_m$ and $e_2 = \alpha A_{m-1} + \beta A_m$. Calculations show that the characteristic polynomial of $D^{(e)}_{n-2}$ is

$$det(D^{(e)}_{n-2} - \lambda I) = -\lambda^{2m-4} r_m(\lambda^2),$$

where the polynomial $r_m(y)$ is defined in Lemma 1. The statement of Lemma 1 completes the proof of Theorem 6.

References

1. J. W. Alexander, 'Functions which map the interior of the unit disc upon simple regions', *Ann. of Math.* 17 (1915-16) 12-22.

2. R. Askey and G. Gasper, 'Positive Jacobi polynomial sums', *Amer. J. Math.* 98 (1976) 709-737.

3. R. Askey and G. Gasper, 'Inequalities for polynomials', *The Bieberbach conjecture: proceedings of the symposium on the occasion of proof*, A. Baernstein II et al., eds. (Amer. Math. Soc., Providence, RI, 1986), pp. 7-32.

4. R. Askey and J. Steinig, 'Some positive trigonometric sums', *Trans. Amer. Math. Soc.* 187 (1974) 295-307.

5. L. de Branges, 'A proof of the Bieberbach conjecture', *Acta Math.* 154 (1985) 137-152.

6. J. Dieudoneé, 'Recherches sur quelques problèmes relatifs aux polynômes et aux fonctions bornées d'une variable complexe', *Ann. Sci. École Norm. Sup.* (3) 48 (1931) 247-358.

7. E. Egerváry and Szász, 'Einige Extremalprobleme im Bereiche der trigonome-trischen Polynome', *Math. Z.* 27 (1928) 641-652.

8. L. Fejér, 'Sur les functions bornées et integrables', *C. R. Acad. Sci. Paris* 131 (1900) 984-987.

9. L. Fejér, 'Über trigonometriche Polynome', *J. Reine Angew. Math.* 146 (1915) 53-82.

10. L. Fejér, 'On new properties of the arithmetical means of the partial sums of Fourier series', *J. Math. Physics* 13 (1934) 83-88.

11. L. Fejér, 'Untersuchungen über Potenzreihen mit mehrfach monotoren Koeffizien-tenfolge', *Litt. Acad. Sci. Szeged* 8 (1936) 89-115.

12. L. FEJÉR AND G. SZEGŐ, 'Special conformal maps', *Duke Math. J.* 18 (1951) 535-548.

13. G. GASPER, 'Nonnegative sums of cosine, ultraspherical and Jacobi polynomials', *J. Math. Anal. Appl.* 26 (1969) 60-68.

14. K. F. GAUSS, 'Développement des fonctions pŕiodiques en séries', Werke VII, (K. Gesell. Wiss., Göttingen, 1906), pp. 469-172.

15. A. GLUCHOFF AND F. HARTMANN, 'Univalent polynomials and non-negative trigonometric sums', *Amer. Math. Monthly* 105 (1998) 508-522.

16. B. L. GOLINSKII, 'On the asymptotic behaviour of the prediction error', *Theory Probab. Appl.* 19 (1974) 693-709.

17. T. H. GRONWALL, 'Über die Gibbssche Erscheinung und die trigonometrischen Summen $\sin x + \frac{1}{2}\sin 2x + \cdots + \frac{1}{n}\sin nx$', *Math. Ann.* 72 (1912) 228-243.

18. E. HEWITT AND R. E. HEWITT, 'The Gibbs-Wilbraham phenomenon: an episode in Fourier analysis', *Arch. Hist. Exact Sci.* 21 (1979) 129-160.

19. G. A. HORN AND C. R. JOHNSON, *Matrix analysis* (Cambridge Univ. Press, Cambridge, 1985).

20. D. JACKSON, 'Über eine trigonometrische Summe', *Rend. Circ. Mat. Palermo* 32 (1911) 257-262.

21. D. JACKSON, 'Über die Genauigkeit der Annäherung stetiger Funktionen durch rationale Funktionen gegebenen Grades und trigonometrische Summen gegebenen Ordnung', doctoral thesis, University of Göttingen, 1911.

22. D. JACKSON, *The theory of approximations*, Amer. Math. Soc. Colloq. Publ., Vol. XI (Providence, R. I.: Amer. Math. Soc., 1930).

23. Y. KATZNELSON, *An introduction to harmonic analysis* (John Wiley, New York, 1968).

24. D. S. LUBINSKY, 'Jackson and Bernstein theorems for exponential weights', *Approximation theory and function series* (Bolyai Society Math. Studies, Budapest, 1996), pp. 86-115.

25. G. V. MILOVANOVIĆ, D. S. MITRINOVIĆ and TH. M. RASSIAS, *Topics in polynomials: extremal problems, inequalities, zeros* (World Scientific, Singapore, 1994).

26. T. J. RIVLIN, *An introduction to the approximation of functions* (Dover Publications, New York, 1981).

27. W. W. ROGOSINSKI and G. SZEGŐ, 'Über die Abschnitte von Potenzreihen, die in einem Kreise beschränkt bleiben', *Math. Z.* 28 (1928) 73-94; reprinted in *Gabor Szegő: Collected papers* Vol. 2, R. Askey, ed., (Birkhäuser, Boston, 1982), pp. 89-110.

28. W. W. ROGOSINSKI and G. SZEGŐ, 'Extremum problems for non-negative sine polynomials', *Acta Sci. Math. Szeged* 12 (1950) 112-124; reprinted in *Gabor Szegő: Collected papers* Vol. 3, R. Askey, ed., (Birkhäuser, Boston, 1982), pp. 196-208.

29. G. SZEGŐ, 'Koeffizientenabschätzungen bei ebenen und räumlichen harmonischen Entwicklungen', *Math. Ann.* 96 (1926/27) 601-632.

30. A. F. TIMAN, *Theory of approximation of functions of a real variable* (Pergammon Press, Oxford, 1963).

31. E. C. TITCHMARSH, *The theory of functions* (Oxford Univ. Press, London, 1939).

32. P. TURÁN, 'On a trigonometric sum', *Ann. Soc. Polon. Math.* 25 (1952) 155-161.

33. L. VIETORIS, 'Über das Vorzeichechen gewisser trigonometrischer Summen', *Sitzungsber. Öst. Akad. Wiss.* 167 (1958) 125-135; *Anzeiger. Öst. Akad. Wiss.* (1959) 192-193.

34. W. VAN ASSCHE, 'Orthogonal polynomials in the complex plane and on the real line', *Fields Inst. Comm.* 14 (1997) 211-245.

35. W. H. YOUNG, 'On certain series of Fourier', *Proc. London Mah. Soc.* (2) 11 (1912) 357-366.

36. A. ZYGMUND, *Trigonometric series* (Cambridge Univ. Press, Cambridge, 1968).

Departamento de Ciências
de Computação e Estatística
IBILCE, Universidade Estadual Paulista
15054-000 São José do Rio Preto
SP, Brazil
E-mail:
dimitrov@nimitz.dcce.ibilce.unesp.br
merlo@nimitz.dcce.ibilce.unesp.br