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1 Inequalities for the zeros of Jacobi polynomials

For any $(\alpha, \beta),\ \alpha, \beta > -1$, denote by $x_{nk}(\alpha, \beta),\ k = 1, \ldots, n$, the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, arranged in decreasing order.

Problem 1 *Let (α, β) be a given pair of real numbers with $\alpha, \beta > -1$. Determine all $(a, b),\ a, b > -1$, such that the inequalities*

$$x_{n,k}(a, b) \ < \ (>) \ x_{n,k}(\alpha, \beta), \quad k = 1, \ldots, n,$$

hold for every positive integer n.

The well-known result of A. Markov [3], [4, Theorem 6.21.1] states that the zeros of $P_n^{(\alpha,\beta)}(x)$ are decreasing functions of α and increasing functions of β. Let (α, β) be fixed. Markov's theorem means that the zeros of $P_n^{(a,b)}(x)$ precede the zeros of $P_n^{(\alpha,\beta)}(x)$ for every n if the vector $(a, b) - (\alpha, \beta)$ is in the fourth quadrant. It is proved in [1] that $x_{n,1}(a, b) < x_{n,1}(\alpha, \beta)$ can be extended to the angle

$$\pi/2 \le \arg((a, b) - (\alpha, \beta)) < 5\pi/4 \qquad (1)$$

and, when $\beta < \alpha$ the same inequalities can be extended to

$$-\pi/2 \le \arg((a, b) - (\alpha, \beta)) < \arg(\alpha + 1, \beta + 1), \qquad (2)$$

except for the points (a,b) in a triangular region Δ. We believe that the largest zero of $P_n^{(a,b)}(x)$ is less than the largest zero of $P_n^{(\alpha,\beta)}(x)$ also when $(a,b) \in \Delta$.

On the other hand, it was proved in [2] that the largest regions where the zeros of $P_n^{(a,b)}(x)$ could precede the zero of $P_n^{(\alpha,\beta)}(x)$ for each natural n, are exactly (1) and (2). Thus we may state the following quite optimistic conjecture: Let $\beta > (<)\alpha$. Then the inequalities

$$x_{n,k}(a,b) < x_{n,k}(\alpha,\beta),\ n = 1,2,\ldots,\quad k = 1,\ldots,n,$$

hold if and only if (a,b) is in (1) (in (2)).

References

[1] D. K. Dimitrov, Connection coefficients and zeros of orthogonal polynomials, available at http:www.dcce.ibilce.unesp.br/ dimitrov.

[2] D. K. Dimitrov and R. O. Rodrigues, On the behaviour of zeros of Jacobi polynomials (submitted).

[3] A. Markov, Sur les racines de certaines équations (second note), *Math. Ann.* **27** (1886) 177–182.

[4] G. Szegő, *Orthogonal Polynomials* 4th ed., Amer. Math. Soc. Coll. Publ., Vol. 23 (Amer. Math. Soc., Providence, RI, 1975).