MARKOV'S INEQUALITY IN L_2 NORM

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ABSTRACT. Denote by π_n the set of algebraic polynomials of degree not exceeding n. Let $\|f\|_2^2 := \int_{-1}^{1} f^2(x)\, dx$. The extremal values

$$M_n(2) := \max\{\|p'\|_2/\|p\|_2 : p \in \pi_n,\ p \neq 0\}$$

in the Markov type inequality in L_2 norm are characterized in terms of the smallest zeros of modified Lommel polynomials. Upper bounds for $M_n(2)$ are given. Moreover, the extremal polynomial for which $M_n(2)$ is attained, is obtained in closed form as a linear combination of the ultraspherical polynomials $P_k^{(-1/2)},\ k = 0, \dots, n$.

1. INTRODUCTION

Let $\|f\|_q := \left(\int_{-1}^{1} |f(x)|^q\, dx\right)^{1/q},\ 1 \leq q < \infty$, and $\|f\|_\infty := \max_{-1\leq x\leq 1} |f(x)|$. In 1889 Markov [7] proved that for every $p \in \pi_n$

$$\|p'\|_\infty \leq n^2\ \|p\|_\infty,$$

and equality is attained if and only if $p(x)$ is a constant multiple of the Chebyshev polynomial $T_n(x)$. In 1937 Hille, Szegő and Tamarkin [4] extended Markov's inequality to the L_q norm, $1 \leq q \leq \infty$. Set

$$M_n(q) := \max\{\|p'\|_q/\|p\|_q : p \in \pi_n,\ p \neq 0\}. \tag{1.1}$$

It was proved in [4] that

$$M_n(q) = c_n(q)n^2, \quad n = 1, 2, \dots ,$$

where $\{c_n(q)\}_{n=1}^{\infty}$ is a bounded sequence for any fixed $q,\ 1 \leq q \leq \infty$. The only known exact values of $c_n(q)$ are that for $q = \infty$. In that case $c_n(\infty) = 1,\ n = 1, 2, \dots$, as it is seen from Markov's result. The problem of finding the precise values of $M_n(q)$ and characterising the extremal polynomials for which $M_n(q)$ is attained for $q,\ 1 \leq q < \infty$, is still open. Particularly interesting is the case $q = 2$. In 1943 Schmidt [8] proved that for any $n \geq 5$

$$M_n(2) = \frac{(n+3/2)^2}{\pi} \frac{1}{1 - \frac{\pi^2-3}{12(n+3/2)^2} + \frac{R_n}{(n+3/2)^4}}, \quad \textit{where} \quad -6 < R_n < 13. \tag{1.2}$$

It follows immediately from (1.2) that $c_n(2) \longrightarrow 1/\pi$ as n diverges, which had been proved earlier by Hille, Szegő and Tamarkin.

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In this work we characterise $M_n(2)$ in terms of the smallest zero $\lambda_{2n,\nu}$ of the modified Lommel polynomial $h_{2n,\nu}$ for some specific values of the parameter ν. On using the explicit expressions for Lommel polynomials we provide a very simple proof of the asymptotic behaviour of $c_n(2)$. Sharp upper bounds for $M_n(2)$ are also obtained. Moreover, the extremal polynomial for which $M_n(2)$ is attained, is represented in closed form as a linear combination of the ultraspherical polynomials $P_k^{(-1/2)}$, $k = 0, \ldots, n$. The coefficients in the linear combination are given in an explicit form in terms of $h_{2k,\nu}(\lambda_{2n,\nu})$, $k = 0, \ldots, n-1$.

Let $P_k, k = 0, 1, \ldots$ be the Legendre polynomilas normalized by $P_k(1) = 1, P_{-1}(x) \equiv 0$. Define the polynomials $\Pi_k, k = 0, 1, \ldots,$ by

$$\begin{aligned} \Pi_0(x) &:= -1/\sqrt{2}, \\ \Pi_1(x) &:= -x/\sqrt{2}, \\ \Pi_k(x) &:= \sqrt{\frac{2k-1}{2}}\frac{1}{k(k-1)}(1-x^2)P'_{k-1}(x), \quad k \geq 2. \end{aligned}$$

The modified Lommel polynomials $h_{n,\nu}(x)$ are generated by the three term recurrence relation (cf. [1, p. 188])

$$h_{0,\nu}(x) = 1, \quad h_{1,\nu}(x) = 2\nu x, \tag{1.3}$$

$$h_{n+1,\nu}(x) = 2(n+\nu)h_{n,\nu}(x) - h_{n-1,\nu}(x), \quad n \geq 1. \tag{1.4}$$

The main reason that the modified Lommel polynomials are significant is that they "connect" Bessel functions of different orders by the relation (cf. Watson [10, Paragraph 9.6])

$$J_{n+\nu}(z) = h_{n,\nu}(1/z)J_\nu(z) - h_{n-1,\nu+1}(1/z)J_{\nu-1}(z).$$

By the Favard theorem [2] (cf. also [1, Theorem 6.4]) $h_{n,\nu}(x)$, $n = 0, 1, \cdots$ form a sequence of orthogonal polynomials with respect to a positive measure for any $\nu > 0$. The explicit orthogonality relation is [1, Chapter 6]

$$\sum_{k=-\infty}^{\infty} \frac{1}{j^2_{\nu-1,k}} h_{m,\nu}(\frac{1}{j_{\nu-1,k}}) h_{n,\nu}(\frac{1}{j_{\nu-1,k}}) = \frac{\delta_{mn}}{2(n+\nu)}, \tag{1.5}$$

where $\{j_{\nu,k}\}_{k=-\infty}^{\infty}$ are the zeros of $z^{-\nu}J_\nu(z)$ arranged as

$$\cdots < j_{\nu,-2} < j_{\nu,-1} < 0 < j_{\nu,1} < j_{\nu,2} < \cdots.$$

Denote by $\lambda_{n,\nu}$ the smallest positive zero of $h_{n,\nu}(x)$.

THEOREM 1. *(i) For every* $p \in \pi_{2n}$

$$\|p'\| \leq \lambda_{2n,3/2}\|p\|,$$

where equality is attained if and only if $p(x)$ is a constant multiple of the polynomial

$$p^*_{2n}(x) = 3\Pi_0(x) + \sum_{k=1}^{n} \sqrt{4k-1}\, h_{2k-2,3/2}(\lambda_{2n,3/2})\Pi_{2k}(x).$$

(ii) For every $p \in \pi_{2n-1}$

$$\|p'\| \leq \lambda_{2n,1/2}\|p\|,$$

where equality is attained if and only if $p(x)$ is a constant multiple of the polynomial

$$p^*_{2n-1}(x) = \sum_{k=1}^{n} \sqrt{4k+1}\, h_{2k-2,1/2}(\lambda_{2n,1/2})\Pi_{2k-1}(x).$$

THEOREM 2. *For any even $n \geq 4$ and for every $p \in \pi_n$*

$$\|p'\| < \frac{n-1/2}{\sin\frac{\pi}{n+2}}\|p\|.$$

For any odd $n \geq 3$ and for every $p \in \pi_n$

$$\|p'\| < \frac{n-1/2}{\sin\frac{\pi}{n+3}}\|p\|.$$

The latter theorem immediately yields:

COROLLARY 1. *For any positive ineteger $n \geq 4$*

$$\|p'\| < \frac{n-1/2}{\sin\frac{\pi}{n+3}}\|p\| \quad \textit{for every} \quad p \in \pi_n.$$

Observe that this statement refines the result of Schmidt (1.2) providing a closed form for an upper bound for $M_n(2)$.

2. Preliminaries

We start this section with two identities for the polynomials $\Pi_n(x)$ (cf. (???) and (???) in [9]):

$$\begin{aligned}
\Pi_k'(x) &= -\sqrt{\frac{2k-1}{2}}P_{k-1}(x), \quad k \geq 1,\\
\Pi_k(x) &= \frac{1}{\sqrt{4k-2}}(P_{k-2}(x) - P_k(x)), \quad k \geq 1.
\end{aligned}$$

Then, identities (???), (???) and the orthogonality of the Legendre polynomials yield

$$\begin{aligned}
\int_{-1}^{1} \Pi_k'(x)\Pi_j'(x)\,dx &= \delta_{kj}\\
\int_{-1}^{1} \Pi_0^2(x)\,dx &= 1\\
\int_{-1}^{1} \Pi_1^2(x)\,dx &= 1/3\\
\int_{-1}^{1} \Pi_k^2(x)\,dx &= \frac{2}{(2k-3)(2k+1)}, \quad k \geq 2 \qquad (2.1)\\
\int_{-1}^{1} \Pi_0(x)\Pi_k(x)\,dx &= -\frac{1}{\sqrt{3}}\delta_{2k}, \quad k \geq 1\\
\int_{-1}^{1} \Pi_k(x)\Pi_j(x)\,dx &= -\frac{1}{\sqrt{2k-1}(2k+1)\sqrt{2k+3}}\delta_{k+2,j}, \quad 1 \leq k < j.
\end{aligned}$$

It is clear that the polynomials $h_{2n,\nu}(x)$ are even functions. The next lemma characterizes the polynomials $h_{2n,\nu}(x)$ in terms of a recurrence relation they satisfy.

LEMMA 1. *Let the sequences of polynomials* $\{R_n^{(e)}(x)\}$ *and* $\{R_n^{(o)}(x)\}$, $n = 0, 1, \dots,$ *be defined by the recurrence relations*

$$\begin{aligned} R_{-1}^{(e)}(x) &= 0 \quad R_0^{(e)}(x) = 1 \\ xR_n^{(e)}(x) &= \frac{1}{\sqrt{4n-1}(4n+1)\sqrt{4n+3}} R_{n-1}^{(e)}(x) + \frac{2}{(4n+1)(4n+5)} R_n^{(e)}(x) \\ &\quad + \frac{1}{\sqrt{4n+3}(4n+5)\sqrt{4n+7}} R_{n+1}^{(e)}(x), \quad n \geq 0, \end{aligned}$$

and

$$\begin{aligned} R_{-1}^{(o)}(x) &= 0 \quad R_0^{(o)}(x) = 1 \\ xR_n^{(o)}(x) &= \frac{1}{\sqrt{4n-3}(4n-1)\sqrt{4n+1}} R_{n-1}^{(o)}(x) + \frac{2}{(4n-1)(4n+3)} R_n^{(o)}(x) \\ &\quad + \frac{1}{\sqrt{4n+1}(4n+3)\sqrt{4n+5}} R_{n+1}^{(o)}(x), \quad n \geq 0. \end{aligned}$$

Then, for any nonnegative integer n, *we have*

$$R_n^{(e)}(x) = \sqrt{\frac{4n+3}{3}} h_{2n,3/2}(\sqrt{x})$$

and

$$R_n^{(o)}(x) = \sqrt{4n+1}\, h_{2n,1/2}(\sqrt{x}).$$

Proof. Observe the renormalized polynomials $\{Q_n^{(e)}(x)\}$ and $\{Q_n^{(o)}(x)\}$,

$$Q_n^{(e)}(x) = (-1)^n \sqrt{\frac{3}{4n+3}} R_n^{(e)}(x), \quad Q_n^{(o)}(x) = (-1)^n \sqrt{4n+1} R_n^{(o)}(x),$$

satisfy the recurrence relations

$$\begin{aligned} Q_0^{(e)}(x) &= 1, \\ Q_1^{(e)}(x) &= 1 - 15x, \\ -xQ_n^{(e)}(x) &= \frac{1}{(4n+3)(4n+5)} Q_{n+1}^{(e)}(x) - \frac{2}{(4n+1)(4n+5)} Q_n^{(e)}(x) \\ &\quad + \frac{1}{(4n+1)(4n+3)} Q_{n-1}^{(e)}(x), n \geq 1 \\ Q_0^{(o)}(x) &= 1, \\ Q_1^{(o)}(x) &= 1 - 3x, \\ -xQ_n^{(o)}(x) &= \frac{1}{(4n+1)(4n+3)} Q_{n+1}^{(o)}(x) - \frac{2}{(4n-1)(4n+3)} Q_n^{(o)}(x) \\ &\quad + \frac{1}{(4n-1)(4n+1)} Q_{n-1}^{(o)}(x), \quad n \geq 1, \end{aligned}$$

It might be mentioned that both $\{Q_n^{(e)}\}$ and $\{Q_n^{(o)}\}$ are sequences of birth and death process polynomials (cf. Karlin and Mc Gregor [6]).

Consider the modified Lommel polynomials $h_{n,\nu}(x)$. Obviously both $h_{2n,\nu}(x)$ and $(1/x)h_{2n+1,\nu}(x)$ are polynomials in x^2 of degree n. Set

$$r_{n,\nu}(x) := (-1)^n h_{2n,\nu}(\sqrt{x}). \tag{2.2}$$

Clearly

$$x(-1)^n r_{n,\nu}(x) = \sqrt{x}\sqrt{x}h_{2n,\nu}(\sqrt{x}) = \sqrt{x}\frac{h_{2n+1,\nu}(\sqrt{x}) + h_{2n-1,\nu}(\sqrt{x})}{2(2n+\nu)}.$$

Therefore

$$(-1)^n 2(2n+\nu)xr_{n,\nu}(x) = \frac{h_{2n+2,\nu}(\sqrt{x}) + h_{2n,\nu}(\sqrt{x})}{2(2n+\nu+1)} + \frac{h_{2n,\nu}(\sqrt{x}) + h_{2n-2,\nu}(\sqrt{x})}{2(2n+\nu-1)}.$$

This implies

$$\begin{aligned}(-1)^n(2n+\nu)xr_{n,\nu}(x) &= \frac{(-1)^{n+1}}{4(2n+\nu+1)}r_{n+1,\nu}(x) + \frac{(-1)^{n-1}}{4(2n+\nu-1)}r_{n-1,\nu}(x)\\ &\quad+\frac{(-1)^n(2n+\nu)}{2(2n+\nu-1)(2n+\nu+1)}r_{n,\nu}(x).\end{aligned}$$

Finally we get

$$\begin{aligned}-xr_{n,\nu}(x) &= \frac{1}{(4n+2\nu)(4n+2\nu+2)}r_{n+1,\nu}(x)\\ &\quad-\frac{2}{(4n+2\nu-2)(4n+2\nu+2)}r_{n,\nu}(x)\\ &\quad+\frac{1}{(4n+2\nu-2)(4n+2\nu)}r_{n-1,\nu}(x). \end{aligned}\tag{2.3}$$

From (1.3), (1.4) and (2.2) we see that $r_{0,\nu}(x) = 1$ and $r_{1,\nu}(x) = -h_{2,\nu}(\sqrt{x}) = 1 - 2(\nu+1)2\nu x$. Thus

$$r_{0,\nu}(x) = 1, \quad r_{1,\nu}(x) = 1 - 2\nu(2\nu+2)x.$$

When $\nu = 3/2$, $r_{1,3/2}(x) = 1 - 15x$ and (2.3) is exactly the recurrence relation for $Q_n^{(e)}(x)$. Therefore $Q_n^{(e)}(x) = r_{n,3/2}(x)$. Similarly $Q_n^{(o)}(x) = r_{n,1/2}(x)$. $\square$

Observe that, since $h_{2m,\nu}(-x) = h_{2m,\nu}(x)$, then (1.5) yiields the following orthonalyty relarion for the polynomials $\{r_{n,\nu}(x)\}$:

$$\sum_{k=1}^{\infty}\frac{1}{j^2_{\nu-1,k}}r_{m,\nu}(\frac{1}{j^2_{\nu-1,k}})r_{n,\nu}(\frac{1}{j^2_{\nu-1,k}}) = \frac{\delta_{mn}}{4(2n+\nu)}.$$

3. Proofs of the main results

Proof of Theorem 1. First identity from (2.2) means that the derivatives of $\Pi_k(x)$ are orthogonal and the remaining one identities show that Π_k are "almost orthogonal". Moreover, the polynomials $\Pi_k,\ k = 0,\dots,n$ form a basis in π_n. Then every polynomial $p(x) \in \pi_n$ can be uniquely represented in the form

$$p(x) = \sum_{k=0}^{n} a_k\Pi_k(x).$$

Then $p'(x) = \sum_{k=1}^{n} a_k\Pi'_k(x)$ and the relations (2.2) yield

$$\|p\|^2 = a_0^2 + \frac{a_1^2}{3} + \sum_{k=2}^{n}\frac{2a_k^2}{(2k-3)(2k+1)} - \frac{2}{\sqrt{3}}a_0a_2 - \sum_{k=1}^{n-2}\frac{2a_ka_{k+2}}{\sqrt{2k-1}(2k+1)\sqrt{2k+3}},$$

and

$$\|p'\|^2 = \sum_{k=1}^{n} a_k^2.$$

Since, by (1.1), $M_n^2(2) := \max\{\|p'\|_2^2/\|p\|_2^2 : p \in \pi_n,\ p \neq 0\}$, then obviously

$$M_n^{-2}(2) := \min\{\|p\|_2^2 \ : \ \|p'\|_2^2 = 1\}. \tag{3.1}$$

Let $S^n := \{ \mathbf{a} = (a_1, \dots, a_n) \in I\!R^n : a_1^2 + \cdots + a_n^2 = 1 \}$ be the unit sphere in $I\!R^n$. Then (3.1) is equivalent to

$$\begin{aligned} M_n^{-2} = \min_{\mathbf{a}\in S^n} \Big\{ & \ a_0^2 + \tfrac{a_1^2}{3} + \textstyle\sum_{k=2}^n \frac{2a_k^2}{(2k-3)(2k+1)} \\ & -\tfrac{2}{\sqrt{3}} a_0 a_2 - \textstyle\sum_{k=1}^{n-2} \frac{2a_k a_{k+1}}{\sqrt{2k-1}(2k+1)\sqrt{2k+3}} \Big\}. \end{aligned}$$

Obviously at the point of minimum $a_0 = a_2/\sqrt{3}$ and then

$$\begin{aligned} M_n^{-2} = \min_{\mathbf{a}\in S^n} \Big\{ & \ a_1^2/3 + a_2^2/15 + \textstyle\sum_{k=3}^n a_k^2 \frac{2}{(2k-3)(2k+1)} \\ & - \textstyle\sum_{k=1}^{n-2} a_k a_{k+2} \frac{2}{\sqrt{2k-1}(2k+1)\sqrt{2k+3}} \Big\}. \end{aligned}$$

It is easy to see that the extremal polynomial is even (odd) if n is even (odd). Hence, for every even n, we have $a_1 = a_3 = \cdots = a_{n-1} = 0$ and for odd n, $a_2 = a_4 = \cdots = a_{n-1} = 0$. Consider first the even case. Setting $b_k = a_{2k}, k = 1, \dots, n$, we obtain

$$\begin{aligned} M_{2n}^{-2} = \min_{\mathbf{b}\in S^n} \Big\{ & \ b_1^2/15 + \textstyle\sum_{k=2}^n b_k^2 \frac{2}{(4k-3)(4k+1)} \\ & - \textstyle\sum_{k=1}^{n-1} b_k b_{k+1} \frac{2}{\sqrt{4k-1}(4k+1)\sqrt{4k+3}} \Big\}. \end{aligned} \tag{3.2}$$

Similarly, setting $b_k = a_{2k-1}, k = 1, \dots, n$, in the odd case, we obtain

$$\begin{aligned} M_{2n-1}^{-2} = \min\nolimits_{\mathbf{b}\in S^n} \Big\{ & \ b_1^2/3 + \textstyle\sum_{k=2}^n b_k^2 \frac{2}{(4k-5)(4k-1)} \\ & - \textstyle\sum_{k=1}^{n-1} b_k b_{k+1} \frac{2}{\sqrt{4k-3}(4k-1)\sqrt{4k+1}} \Big\}. \end{aligned} \tag{3.3}$$

Observe now that (3.2) and (3.3) can be rewritten in the forms

$$M_{2n}^{-2} = \min_{<\mathbf{b},\mathbf{b}>=1} < \mathbf{b}, C_n^{(e)}\mathbf{b} >= \min \frac{< \mathbf{b}, C_n^{(e)}\mathbf{b} >}{< \mathbf{b}, \mathbf{b} >}$$

and

$$M_{2n-1}^{-2} = \min_{<\mathbf{b},\mathbf{b}>=1} < \mathbf{b}, C_n^{(o)}\mathbf{b} >= \min \frac{< \mathbf{b}, C_n^{(o)}\mathbf{b} >}{< \mathbf{b}, \mathbf{b} >},$$

where $< \cdot, \cdot >$ is the inner product in $I\!R^n$ and $C_n^{(e)}$ and $C_n^{(o)}$ are the tridiagonal Jacobian positive definite matrices defined as follows: The diagonal entries of $C_n^{(e)}$ are

$$a_{11}^{(e)} = 1/15,\ a_{kk}^{(e)} = \frac{2}{(4k-3)(4k+1)},\ k = 2, \dots, n$$

and its off diagonal elements are

$$a_{k,k+1}^{(e)} = a_{k+1,k}^{(e)} = -\frac{1}{\sqrt{4k-1}(4k+1)\sqrt{4k+3}},\ k = 1, \dots, n-1.$$

Similarly, the diagonal entries of $\{A_n^{(o)}\}$ are

$$a_{11}^{(o)} = 1/3,\ a_{kk}^{(o)} = \frac{2}{(4k-5)(4k-1)},\ k = 2, \dots, n$$

and its off diagonal elements are

$$a_{k,k+1}^{(o)} = a_{k+1,k}^{(o)} = -\frac{1}{\sqrt{4k-3}(4k-1)\sqrt{4k+1}},\ k = 1, \dots, n-1.$$

Let $\mu_n^{(e)}$ and $\mu_n^{(o)}$ be the smallest eigenvalues of $C_n^{(e)}$ and of $C_n^{(o)}$, respectively, and $\mathbf{b}_*^{(e)}$, $\mathbf{b}_*^{(o)} \in I\!R^n$ be the corresponding eigenvectors. By the Rayleigh-Ritz Theorem (cf. Horn and Johnson [5, p. 176]),

$$\min \frac{<\mathbf{b}, C_n^{(e)}\mathbf{b}>}{<\mathbf{b},\mathbf{b}>} = \mu_n^{(e)} \quad \textit{and} \quad \min \frac{<\mathbf{b}, C_n^{(o)}\mathbf{b}>}{<\mathbf{b},\mathbf{b}>} = \mu_n^{(o)},$$

and the minimums are attained if and only if $\mathbf{b}$ coincide with the eigenvectors $\mathbf{b}_*^{(e)}$ and $\mathbf{b}_*^{(o)}$.

It is easily seen that the characteristic polynomials of $C_n^{(e)}$ and of $C_n^{(e)}$ coincide with the polynomials $\{R_n^{(e)}(x)\}$ and $\{R_n^{(o)}(x)\}$ defined in Lemma 1. Obviosusly the smallest eigenvalues $\mu_n^{(e)}$ and $\mu_n^{(o)}$ coincide with the smallest zeros $\zeta_n^{(e)}$ of $R_n^{(e)}$ and $\zeta_n^{(o)}$ of $R_n^{(o)}$. Moreover, an eigenvector of $C_n^{(e)}$, corresponding to $\mu_n^{(e)}$ is

$$\mathbf{b}_*^{(e)} = \left(R_0^{(e)}(\zeta_n^{(e)}), R_1^{(e)}(\zeta_n^{(e)}), \cdots, R_{n-1}^{(e)}(\zeta_n^{(e)})\right),$$

and similarly, an eigenvector of $C_n^{(o)}$, corresponding to $\mu_n^{(o)}$ is

$$\mathbf{b}_*^{(o)} = \left(R_0^{(o)}(\zeta_n^{(o)}), R_1^{(o)}(\zeta_n^{(o)}), \cdots, R_{n-1}^{(o)}(\zeta_n^{(o)})\right).$$

Thus, we conclude:

(i) for every $p \in \pi_{2n}$

$$\|p'\|^2 \ / \ \|p\|^2 \ \leq \ 1 \ / \ \zeta_n^{(e)},$$

where the equality is attained if and only if

$$p(x) = \text{const} \ \left\{\sqrt{3}\Pi_0(x) + \sum_{k=1}^{n} R_{k-1}^{(e)}(\zeta_n^{(e)})\Pi_{2k}(x)\right\},$$

(ii) for every $p \in \pi_{2n-1}$

$$\|p'\|^2 \ / \ \|p\|^2 \ \leq \ 1 \ / \ \zeta_n^{(o)},$$

where the equality is attained if and only if

$$f(x) = \text{const} \sum_{k=1}^{n} R_{k-1}^{(o)}(\zeta_n^{(o)})\Pi_{2k-1}(x).$$

Now Lemma 1 comletes the proof of Theorem 1.

Proof of Theorem 2. Recall that $M_{2n}^{-2} = \zeta_n^{(e)}$ and $M_{2n-1}^{-2} = \zeta_n^{(o)}$ and, because of Lemma 1, $\zeta_n^{(e)}$ and $\zeta_n^{(o)}$ are the smallest zeros of $r_{n,3/2}(x)$ and of $r_{n,1/2}(x)$. The theorem will be proved if we prove that, for $\nu = 1/2$ and $\nu = 3/2$, the smallest zeros of $r_{n,\nu}(x)$ are greater than

$$\frac{\sin^2 \frac{\pi}{2(n+1)}}{(2n+\nu-2)^2}.$$

In view of Therems 1 and 3 in [3], all we need to show, is that for the above values of ν, the inequalities

$$\begin{aligned}
\frac{\sin^2 \frac{\pi}{2(n+1)}}{(2n+\nu-2)^2} \quad &< \quad \frac{1}{4\nu(\nu+1)}, \\
\frac{\sin^2 \frac{\pi}{2(n+1)}}{(2n+\nu-2)^2} \quad &< \quad \frac{1}{2(2k+\nu-1)(2k+\nu+1)}, \quad k = 1, \ldots, n-1, \qquad (3.4)
\end{aligned}$$

and

$$\left(\frac{\sin^2\frac{\pi}{2(n+1)}}{(2n+\nu-2)^2}-\frac{1}{4\nu(\nu+1)}\right)\times$$
$$\left(\frac{\sin^2\frac{\pi}{2(n+1)}}{(2n+\nu-2)^2}-\frac{1}{2(\nu+1)(\nu+3)}\right)>$$
$$\frac{\cos^2\frac{\pi}{n+1}}{4(2n+\nu-2)(2n+\nu-1)^2(2k+\nu)},$$

$$\left(\frac{\sin^2\frac{\pi}{2(n+1)}}{(2n+\nu-2)^2}-\frac{1}{2(2k+\nu-1)(2k+\nu+1)}\right)\times$$
$$\left(\frac{\sin^2\frac{\pi}{2(n+1)}}{(2n+\nu-2)^2}-\frac{1}{2(2k+\nu-3)(2k+\nu-1)}\right)>$$
$$\frac{\cos^2\frac{\pi}{n+1}}{4(2n+\nu-2)(2n+\nu-1)^2(2k+\nu)},\quad k=2,\dots,n-1, \tag{3.5}$$

hold.

We shall prove that (3.4) and (3.5) hold for every positive integer $n\geq 2$ and for any real $\nu, 0<\nu<3$. In order to this, let us introduce the denotations

$$\begin{aligned}
\beta_0 &:= \frac{1}{4\nu(\nu+1)},\\
\beta_k &:= \frac{1}{2(2k+\nu-1)(2k+\nu+1)},\quad k\geq 1,\\
x_k &:= \frac{\sin^2\frac{\pi}{2(n+1)}}{(2k+\nu)^2},\quad k\geq 1,\\
g_k &:= \frac{\sin^2\frac{\pi}{2(n+1)}}{4(2k+\nu-2)(2k+\nu-1)^2(2k+\nu)},\quad k\geq 1.
\end{aligned}$$

Observe that $\beta_1<\beta_0$ for $0<\nu<3$. Then, obviously the sequence $\{\beta_k\}_{k=0}^{n-1}$ is a decreasing one. It is clear that $\{x_k\}_{k=0}^{n-1}$ is also a decreasing sequence for $\nu>0$. Note first that the inequalities

$$x_k<\beta_k,\quad k=1,\dots,n-1 \tag{3.6}$$

follow immediately from the obvious ineqiualities $\sin^2(\pi/(2n+2))\leq 1/2$ and $y^2/((y-1)(y+1))>1$ with $y=2k+\nu-2$. Now (3.6) and the fact that $\{\beta_k\}_{k=0}^{n-1}$ and $\{x_k\}_{k=0}^{n-1}$ are decreasing yield the inequalities

$$x_{n-1}<\beta_k,\quad k=0,\dots,n-1,\quad \text{for every} n\geq 2 \text{ and } 0<\nu<3,$$

which are equivalent to (3.4).

Set

$$f_k(x):=(x-\beta_{k-1})(x-\beta_k).$$

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$$\begin{aligned}
f_k(x_k) &= (x_k-\beta_{k-1})(x_k-\beta_k)\\
&= \left(\frac{\sin^2\frac{\pi}{2(n+1)}}{(2k+\nu)^2}-\frac{1}{2(2k+\nu-1)(2k+\nu+1)}\right)\\
&\quad\times\left(\frac{\sin^2\frac{\pi}{2(n+1)}}{(2k+\nu)^2}-\frac{1}{2(2k+\nu-3)(2k+\nu-1)}\right)\\
&= \frac{1}{4(2k+\nu-3)(2k+\nu-1)^2(2k+\nu+1)}\\
&\quad\times\left\{1-c_k2\sin^2\frac{\pi}{2(n+1)}\right\}\left\{1-d_k2\sin^2\frac{\pi}{2(n+1)}\right\},
\end{aligned}$$

where

$$c_k=\frac{(2n+\nu)^2-1}{(2n+\nu)^2},\quad\text{and}\quad d_k=\frac{(2n+\nu-2)^2-1}{(2n+\nu)^2}.$$

Since $c_k<1$ and $d_k<1$, then the two expressions in the curly brackets are greater than $1-2\sin^2(\pi/(2n+2))=\cos(\pi/(n+1))$. Thus

$$f_k(x_k)>\frac{\cos^2\frac{\pi}{n+1}}{4(2k+\nu-3)(2k+\nu-1)^2(2k+\nu+1)}>g_k,$$

where the latter inequality is equivalent to $y^2-4<y^2-1$ with $y=2k+\nu-1$. □

3. Assymptotics of the extremal values

In this section we provide a very simple proof of the assymptotics $c_n(2)\to 1/\pi$, which, as was already mentioned, was proved in different ways by Hille, Szegő and Tamarkin [4] and by Schmidt [8]. In view of Theorem 1, all we need to show is that the smallest zeros of $r_{n,1/2}(x)$ and of $r_{n,3/2}(x)$ are of the form $d_n n^{-4}$, where $d_n\to\pi^2/16$ as n diverges. In order to this, note that (cf. ???)

$$h_{m,\nu}(x)=\sum_{0\le j\le m/2}\frac{(-1)^j(m-j)!(\nu)_{m-j}}{j!(\nu)_j}\frac{(2x)^{m-2j}}{(m-2j)!},$$

where $(a)_k$ denotes the Pochhammer symbol, $(a)_k=a(a+1)\cdots(a+k-1)$ with $(a)_0=1$. It is clear that $(a)_k=\Gamma(a+k)/\Gamma(a)$. Observe that

$$(-1)^n h_{2n,\nu}(x)=\sum_{j=0}^{n}\frac{(2n-j)!\Gamma(2n+\nu-j)}{\Gamma(\nu+j)}\frac{(-1)^{n-j}(2x)^{2n-2j}}{j!(2n-2j)!}.$$

Replace j by $n-k$ and x by $\sqrt{x}$ to obtain

$$r_{n,\nu}(x)=\sum_{j=0}^{n}\frac{\Gamma(n+k+1)\Gamma(n+k+\nu)}{\Gamma(n-k+1)\Gamma(n-k+\nu)}\frac{(-4x)^k}{(2k)!},$$

where we used $m! = \Gamma(m+1)$. Since $x^{b-a}\Gamma(x+a)/\Gamma(x+b) \longrightarrow 1$ as $x \to +\infty$, one can easily show that

$$\begin{aligned} r_{n,\nu}(\frac{x}{n^4}) &= \sum_{j=0}^{n} \frac{\Gamma(n+k+1)}{\Gamma(n-k+1)} n^{-2k} \frac{\Gamma(n+k+\nu)}{\Gamma(n-k+\nu)} \frac{(-4x)^k}{(2k)!} \\ &\to \sum_{k=0}^{\infty} \frac{(-4x)^k}{(2k)!} \quad \text{as} \quad n \to \infty. \end{aligned}$$

This implies that the smallest zero of $r_{n,\nu}(x)$ is of the form $d_n n^{-4}$ and the limit of the sequence $\{d_n\}$ is equal to the smallest zero of $\cos(2\sqrt{x})$, i.e. $\{d_n\} \to \pi^2/16$. It is worth observing that if the zeros of $r_{n,\nu}(x)$ are $0 < x_{n,1} < \ldots < x_{n,n}$, then $n^4 x_{n,1} \to \pi/16$ for all $\nu > 0$, that is the latter convergence is independent of ν.

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