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GAUSSIAN EXTENDED CUBATURE FORMULAE FOR POLYHARMONIC FUNCTIONS

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ABSTRACT. The purpose of this paper is to show certain links between univariate interpolation by algebraic polynomials and the presentation of polyharmonic functions. This allows us to construct cubature formulae for multivariate functions having highest order of precision with respect to the class of polyharmonic functions. We obtain a Gauss type cubature formula that uses m values of linear functionals (integrals over hyperspheres) and is exact for all $2m$-harmonic functions, and consequently, for all algebraic polynomials of n variables of degree $4m-1$.

1. Introduction and statement of results

Let $I\!R^n$ be the real n-dimensional Euclidean space. The points of $I\!R^n$ are denoted by $x = (x_1, x_2, \ldots, x_n)$ and $|x|$ is the Euclidean norm of x, that is, $|x| := \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. For any positive r, we denote by $B(r)$, $\bar{B}(r)$ and $S(r)$ the open and the closed balls and the hypersphere with center 0 and radius r in $I\!R^n$. Precisely,

$$\begin{aligned} B(r) &:= \{x \,:\, |x| < r\}, \\ S(r) &:= \{x \,:\, |x| = r\}, \\ \bar{B}(r) &:= B(r) \cup S(r). \end{aligned}$$

In case $r = 1$ we shall omit the notation of the radius. The outside normal derivative on S is denoted by $\frac{\partial}{\partial \nu}$. Finally, dx is Lebesgue measure in $I\!R^n$ and $d\sigma$ is the $(n-1)$-dimensional surface measure on $S(r)$. Recall that the area $\sigma_n(r)$ of the sphere $S(r)$ in $I\!R^n$ is

$$\sigma_n(r) = n r^{n-1} \pi^{n/2} / \Gamma(n/2 + 1)$$

where Γ is the Gamma function.

The iterates Δ^m of the Laplace operator in $I\!R^n$ are defined recursively by

$$\begin{aligned} \Delta &= \Delta^1 = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}, \\ \Delta^m &= \Delta \Delta^{m-1}. \end{aligned}$$

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The function u is said to be *polyharmonic* of order m, or *m-harmonic*, in B if u belongs to the space

$$H^m(B) := \left\{ u \in C^{2m-1}(\bar{B}) \bigcap C^{2m}(B) \quad : \quad \Delta^m u = 0 \;\; \text{on} \;\; B \right\}.$$

In particular, if $m = 1$ or $m = 2$, u is said to be *harmonic* or *biharmonic*, respectively.

Being null spaces of the even-order differential operator Δ^m, the polyharmonic functions of order m inherit many of the properties of the univariate algebraic polynomials of odd degree $2m - 1$. Some recent developments reveal the importance of polyharmonic functions as an appropriate tool in multivariate approximation. Classical results in approximation theory have been extended to theorems treating approximation of multivariate functions by m-harmonic functions (see [4], [1], for example, and the references in the papers therein which contain such results). The purpose of this paper is to show certain links between univariate interpolation by algebraic polynomials and the representation of polyharmonic functions. We give natural polyharmonic analogues of the Lagrange and Newton representations of algebraic polynomials. This allows us to construct cubature formulae for multivariate functions having highest degree of precision with respect to the class of polyharmonic functions. We obtain a Gauss type cubature formula that uses m values of linear functionals (integrals over hyperspheres) and is exact for all $2m$-harmonic functions, and consequently, for all algebraic polynomials of n variables of degree $4m-1$. In order to formulate this result precisely, let us first recall some definitions from [3] where polyharmonic extensions of other well-known quadrature rules have been obtained.

Every linear functional $Q(f)$ approximating the integral $I(f) = \int_B f(x)\,dx$ in terms of values of $\Delta^i f,\ i = 0, \dots,$ at certain points and/or surface integrals of them and their normal derivatives is called an *extended cubature formula* or *extended cubature rule*. An extended formula is said to have *polyharmonic order of precision* m (indicated briefly as $PHOP(Q) = m$), if $I(f) = Q(f)$ for all $f \in H^m(B)$ and there exists a function f such that $\Delta^m f \not\equiv 0$ in B and $I(f) \neq Q(f)$.

Typical examples of extended cubature formulae are the Gaussian mean-value property

$$\int_{B(r)} u(x)\,dx = r^n \left[\pi^{n/2}/\Gamma(n/2+1)\right] \; u(0) \tag{1.1}$$

and the following consequence of the first Green formula

$$\int_{B(r)} u(x)\,dx \;=\; r/n \int_{S(r)} u(\xi)\,d\sigma(\xi) \tag{1.2}$$

which hold for every harmonic function $u(x)$. They can be looked upon as striking multivariate analogues of the midpoint and of the trapezoid quadrature formulae, respectively.

Note here the following immediate consequence of the above formulae which also holds for every harmonic function $u(x)$ and will be frequently used in the sequel.

$$\int_{S(r)} u(\xi)\,d\sigma(\xi) = \gamma_n r^{n-1} u(0), \qquad \gamma_n := n\pi^{n/2}/\Gamma(n/2+1). \tag{1.3}$$

A univaruate function $\mu(t)$, defined on the interval $[a,b]$, is said to be an weight function on $[a,b]$ if $\mu(t)$ is non-negative there and all the moments $\int_a^b t^s \mu(t)dt$, $s = 0, 1, \dots$ exist.

In this paper we prove the following.

THEOREM 1. *Let $\mu(t)$ be any given weight function on $[0,1]$. There exists a unique sequence of distinct radii $0 < R_1 < \cdots < R_m \leq 1$ and real weights A_k, $k = 1, \cdots, m$, such that the extended cubature formula*

$$\int_B u(x)\ \mu(|x|)dx \ \approx\ \sum_{k=1}^{m} A_k \int_{S(R_k)} u(\xi)\, d\sigma(\xi) \tag{1.4}$$

has polyharmonic order of precision $2m$. Moreover, the radii R_k coincide with the positive zeros of the polynomial $P_{2m}(t;\mu^*)$ of degree $2m$, which is orthogonal on $[-1,1]$ with respect to the weight function $\mu^*(t) = |t|^{n-1}\mu(|t|)$ to any polynomial of degree $2m-1$.

There is no extended cubature formula of the form (1.4) with $PHOP > 2m$.

The coefficients $\{A_k\}$ are explicitly determined as integrals of univariate polynomials and they are positive.

Formula (1.4) can be considered as a polyharmonic extension of the relation (1.2). The problem of extending the Gaussian quadrature formula to the multivariate setting has been of constant interest. The central concept was developed in an attempt to answer the following natural question: Are there ordinary cubature formulae (linear combinations of values of the integrand at certain points) of highest possible total algebraic degree of precision and what is the relation between the number of the values of the integrand involved and the highest possible algebraic degree of precision? Although upper and lower bounds are known, and a few explicit examples are constructed, there are still many open questions in this difficult domain of ordinary cubature formulae. The above stated result solves the problem of existence and uniqueness of extended cubature formulae for integrals over balls in $\mathbb{R}^n$.

Note that the Gaussian extended cubature formula (1.4) approximates the integral of f over B in terms of m pieces of information about the integrand, namely, the "average" values of f over m hyperspheres. It integrates exactly all polyharmonic functions of order $2m$. The space $H^{2m}(B)$ obviously contains the class $\pi_N(\mathbb{R}^n)$ of all algebraic polynomials of n variables of total degree $N := 4m-1$. Thus the extended cubature (1.4) is precise for any polynomial from $\pi_N(\mathbb{R}^n)$ and particularly for the basic polynomials there. Note that, if $n > 1$, the number $dim\pi_N(\mathbb{R}^n)$ of these polynomials is essentially greater than the number m of pieces of information (the spherical integrals) the cubature unvolves. The classical Gaussian quadrature formula integrates exactly the polynomials in the space $\pi_{2m-1}(\mathbb{R})$, whose dimention is only two times bigger than the number m of the function values the quadrature uses. It is shown in the proof of Theorem 1 that (1.4) is the cubature of highest algebraic degree of precision of the type considered. Cubature formulae which are precise for algebraic polynomials have been intensively studied (see for example [8], where the results of Kantorovich and Lusternik concerning algebraic degree of precision of formulae of type (1.4) are given). We show here that some of these known polynomial type formulae integrate exactly the wider class of polyharmonic functions.

The problem of extending Hermite interpolation formulae and multiple node Gaussian quadratures is also considered in Section 5.

2. Representations of polyharmonic functions

In this section we discuss various representations of an m-harmonic function which are induced by univariate polynomial bases. In particular, we get representations that are similar to the Lagrange and Newton-Gregory interpolating polynomials.

Our results are based on the following fundamental result in the theory of polyharmonic functions, known as *Almansi's expansion* (see [2]).

Lemma 1. *If $u \in H^m(B)$, then there exist unique functions $h_0(x), h_1(x), \dots, h_{m-1}(x)$, each harmonic in B, such that*

$$u(x) = \sum_{j=0}^{m-1} |x|^{2j} h_j(x) \quad \textit{for} \quad x \in B. \tag{2.1}$$

It is also known that any expression of the form (2.1), with harmonic functions h_j on B, is a m-harmonic function on B. Thus (2.1) exactly describes the structure of the m-harmonic functions. Based on Almansi's expansion and since harmonic functions in B form a linear space, the following lemma concerning the representation of m-harmonic functions easily holds.

Denote by π_m the set of algebraic polynomials of one variable, of degree less than or equal to m.

Lemma 2. *Let $\phi_0(t), \cdots, \phi_{m-1}(t)$ be any basis in the space π_{m-1} of univariate algebraic polynomials of degree not exceeding $m-1$. If $u \in H^m(B)$, then there exist unique functions $b_0(x), b_1(x), \cdots, b_{m-1}(x)$, each harmonic in B, such that*

$$u(x) = \sum_{j=0}^{m-1} \phi_j(|x|^2) b_j(x) \quad \textit{for} \quad x \in B. \tag{2.2}$$

As an immediate consequence of Lemma 2 and the of fact that the polynomials $1, t-R^2, (t-R^2)^2, \cdots, (t-R^2)^{m-1}$ constitute a basis in π_{m-1} we obtain

Corollary 1. *If $u \in H^m(B)$, then there exist unique functions $b_0(x), b_1(x), \cdots, b_{m-1}(x)$, each harmonic in B, such that*

$$u(x) = \sum_{j=0}^{m-1} (|x|^2 - R^2)^j b_j(x) \quad \textit{for} \quad x \in B. \tag{2.3}$$

The latter representation is related to a recent result of Hayman and Korenblum [6, Theorem 1].

Let us mention another representation which could be of some interest.

Let f be a given univariate function, defined at the distinct points t_i, $i = 1, \dots, m$. In what follows the set of points $t_1, \dots, t_m$ will be denoted by T. The unique algebraic polynomial of degree $m-1$ which interpolates f at $t_1, \dots, t_m$ can be represented in the form

$$L_{m-1}(f; t) = \sum_{k=1}^{m} l_k(t; T) f(t_k),$$

where the basic polynomials $l_k(t;T),\ k=1,\dots,m$, are given by

$$l_k(t;T)=\omega(t)/((t-t_k)\omega'(t_k))$$

in terms of the polynomial $\omega(t):=(t-t_1)\cdots(t-t_m)$.

COROLLARY 2. *Let $0<t_1<t_2<\cdots<t_m\le 1$ be any set of distinct points. Then for any $u\in H^m(B)$ there exist unique harmonic functions $b_1(x),b_2(x),\cdots,b_m(x)$, each harmonic in B, such that*

$$u(x)=\sum_{k=1}^{m} l_k(|x|^2;T)b_k(x)\quad for\quad x\in B.$$

Moreover, $b_k(x)$ is the unique harmonic function on B which coincides with u on $S(\sqrt{t_k})$.

This is the Lagrangean type representation of polyharmonic functions in terms of traces over $(n-1)$-dimensional spheres.

Note here that the same statement holds in the slightly more general case allowing $t_1=0$. Then $b_1(0)=u(0)$ but this condition does not characterise b_1 completely. We shall mention however that the condition $b_1(0)=u(0)$ defines the integral $\int_{S(t)} b(\xi)\ d\sigma(\xi)$ uniquely (because of (1.3)) and this is what is important in the study of integration problems.

Similarly, starting from the Newton-Gregory interpolation formula

$$L_{m-1}(f;t)=\sum_{k=0}^{m-1} f[t_0,\dots,t_k](t-t_0)\cdots(t-t_{k-1})$$

with $0<t_0<\cdots<t_{m-1}\le 1$, one can get the Newton type representation of polyharmonic functions

$$u(x)=\sum_{k=0}^{m-1}(|x|^2-t_0)\cdots(|x|^2-t_{k-1})\,b_k(x).$$

3. GAUSSIAN CUBATURE FORMULAE

In this section we give a simple relation between the univariate quadrature formula for algebraic polynomials and extended cubature formulae for polyharmonic functions.

We shall use the mean value

$$\frac{1}{\gamma_n t^{n-1}}\int_{S(t)} f(\xi)\ d\sigma(\xi)$$

as a main piece of information for approximate evaluation of the integral of f over B. Let us accept here the convention to use the same notation in the case $t=0$ as well. Then clearly the mean value is just $f(0)$.

LEMMA 3. *Assume that $\mu(t)$ is a fixed weight function on $[0,1]$. Let $0\le t_1<\cdots<t_N\le 1$. The extended cubature formula*

$$\int_B \mu(|x|)f(x)dx\approx\sum_{k=1}^{N} a_k\,\frac{1}{\gamma_n t_k^{n-1}}\int_{S(t_k)} f(\xi)\ d\sigma(\xi) \tag{3.1}$$

is exact for every polyharmonic function $f \in H^m(B)$ if and only if the quadrature formula

$$\gamma_n \int_0^1 \mu(t) t^{n-1} P(t^2)\, dt \approx \sum_{k=1}^{N} a_k P(t_k^2) \tag{3.2}$$

is exact for every algebraic polynomial $P \in \pi_{m-1}$.

Proof. Let f be any polyharmonic function of order m. Then, by Almansi's representation, there exist harmonic functions $\{h_k\}$ in B such that

$$f(x) = \sum_{k=0}^{m-1} |x|^{2k} h_k(x).$$

Set

$$P_f(\tau) := \sum_{k=0}^{m-1} h_k(0)\, \tau^k.$$

We have

$$\begin{aligned} \int_B \mu(|x|) f(x)\, dx &= \sum_{k=0}^{m-1} \int_B \mu(|x|) h_k(x) |x|^{2k}\, dx \\ &= \sum_{k=0}^{m-1} \int_0^1 \int_{S(t)} \mu(|\xi|) h_k(\xi) |\xi|^{2k}\ d\sigma(\xi)\, dt \\ &= \sum_{k=0}^{m-1} \int_0^1 \mu(t) t^{2k} \int_{S(t)} h_k(\xi)\ d\sigma(\xi)\, dt \end{aligned}$$

Let us apply now the relation (1.3) to the spherical integral. We get

$$\int_B \mu(|x|) f(x)\, dx = \gamma_n \sum_{k=0}^{m-1} \int_0^1 \mu(t) t^{2k} t^{n-1} h_k(0)\, dt$$

and therefore

$$\int_B \mu(|x|) f(x)\, dx = \gamma_n \int_0^1 \mu(t) t^{n-1} P_f(t^2)\, dt. \tag{3.3}$$

On the other hand, by (1.3),

$$\begin{aligned} \sum_{k=1}^{N} a_k \frac{1}{\gamma_n t_k^{n-1}} \int_{S(t_k)} f(\xi)\ d\sigma(\xi) &= \sum_{k=1}^{N} a_k \frac{1}{\gamma_n t_k^{n-1}} \int_{S(t_k)} \sum_{j=0}^{m-1} |\xi|^{2j} h_j(\xi)\ d\sigma(\xi) \\ &= \sum_{k=1}^{N} a_k \sum_{j=0}^{m-1} t_k^{2j} h_j(0) \\ &= \sum_{k=1}^{N} a_k P_f(t_k^2). \end{aligned}$$

The assertion of the lemma then follows by comparison of the both sides of (3.3) and the last equality. □

The next theorem is an immediate consequence of Lemma 3.

THEOREM 2. *Let $\mu(t)$ be a given weight function on $[0,1]$. If the coefficients a_k, $k = 1,\dots,m$, of the extended cubature formula*

$$\int_B \mu(|x|)f(x)\,dx \approx \sum_{k=1}^{m} a_k \frac{1}{\gamma_n t_k^{n-1}} \int_{S(R_k)} f(\xi)\; d\sigma(\xi) \tag{3.4}$$

are given by

$$a_k = \frac{\gamma_n}{R_k^{n-1}} \int_0^1 l_k(t^2; R_1^2, \cdots, R_m^2)\mu(t)t^{n-1}dt, \tag{3.5}$$

with some $0 \le R_1 < \cdots < R_m \le 1$, then it is precise for every $f \in H^m(B)$. Conversely, if an extended cubature rule of the form (3.4) *is precise for each $f \in H^m(B)$, then its coefficients are uniquely determined by* (3.5).

For the proof, one needs to observe only that the coefficient a_k can be found just by applying the corresponding univariate quadrature formula (given as in (3.2)) to the function $l_k(t^2; R_1^2, \cdots, R_m^2)$.

Now we can give the proof of Theorem 1.

Proof of Theorem 1. In view of Lemma 3, the only thing we need to do is to characterise all quadrature formulae of form (3.2) that are exact for all even algebraic polynomials of degree $4m-2$ (i.e., for all $P \in \pi_{2m-1}$). Assume that (3.2) is such a formula. Then the quadrature

$$\int_{-1}^{1} \mu(|t|)|t|^{n-1}Q(t)\,dt \approx \sum_{k=1}^{N} a_k |t_k|^{n-1}\Big[Q(t_k) + Q(-t_k)\Big]$$

will be exact for all $Q \in \pi_{4m-1}$ since it integrates all even polynomials of degree $4m-1$ by assumption and all odd ones, by construction. Note that it is based on $2N$ nodes. For $N = m$ it must coincide with the Gaussian quadrature formula on $[-1,1]$ corresponding to the weight $\mu(|t|)|t|^{n-1}$. Since the weight is symmetric, then the nodes are symmetric too and moreover, they are located at the zeros of the polynomial of degree $2m$ which is orthogonal on $[-1,1]$ to all polynomials of degree $2m-1$ with respect to the weight μ^*.

The positivity of the coefficients follows from the corresponding property in the univariate case. The proof is completed. ▯

COROLLARY 3. *Let*

$$P_{2m}(|x|,\mu^*) = \prod_{k=1}^{m}(|x|^2 - R_k^2)$$

be defined as in Theorem 1, i.e., let R_k, $k = 1,\dots,m$, be the radii of the Gaussian extended cubature formula (1.4). *Then $P_{2m}(|x|,\mu^*)$ is orthogonal on B with respect to the weight function $\mu(|x|)$ to every harmonic function of order m.*

Proof. We need to show that

$$\int_B P_{2m}(|x|,\mu^*)u_m(x)\mu(|x|)dx = 0 \;\; \textit{for every} \;\; u_m \in H^m(B).$$

Expressing $u_m(x)$ in terms of its Almansi's expansion, one concludes that the product $P_{2m}(|x|,\mu^*)u_m(x)$ is in $H^{2m}(B)$. Then an application of (1.4) to this product and the fact that the radial polynomial $P_{2m}(|x|,\mu^*)$ vanishes on the hyperspheres $S(R_k)$ yield the desired orthogonal property. ▯

It is clear how Radau and Lobatto type extended cubature formulae

$$\int_B \mu(|x|) f(x)\, dx \approx \sum_{k=1}^{m} C_k \int_{S(R_k)} f(\xi)\, d\sigma(\xi) + B_0 f(0),$$

$$\int_B f(x)\mu(|x|)\, dx \approx \sum_{k=1}^{m} C_k \int_{S(R_k)} f(\xi)\, d\sigma(\xi) + B_0 f(0) + B_1 \int_S f(\xi)\, d\sigma(\xi)$$

of highest polyharmonic order of precision can be constructed. The free "nodes" $R_1, \dots, R_m$ are determined as the positive zeros of the polynomial P_{2m} of degree $2m$ which is orthogonal on $[-1, 1]$ with respect to the weight functions $t^2\mu(|t|)|t|^{n-1}$, and $t^2(1-t^2)\mu(|t|)|t|^{n-1}$, respectively, to all polynomials of degree $2m-1$. We omit the details.

4. An extremal problem

Similarly to the univariate case the problem of least integral of positive expressions of the form $f(x) = |x|^{4m} + u(x)$ where u is $2m$ -harmonic can be solved using the extended Gaussian theorem.

Theorem 3. *Among all non-negative functions of the form* $f(x) = |x|^{4m} + u(x)$, $u \in H^{2m}(B)$, *the function*

$$\Omega^*(x) := \prod_{k=1}^{m} (|x|^2 - R_k^2)^2,$$

where $\{R_k\}$ *are the Gaussian radii, has the minimal integral over* B.

The proof goes as in the univariate case and uses the positivity of the Gaussian coefficients. Since $f(x) - \Omega^*(x)$ belongs to $H^{2m}(B)$ the extended Gaussian cubature integrates this function exactly and thus we have

$$\begin{aligned}\int_B (f(x) - \Omega^*(x))dx &= \sum_{k=1}^{m} A_k \int_{S(R_k)} [f(x) - \Omega^*(x)]\, d\sigma \\ &= \sum_{k=1}^{m} A_k \int_{S(R_k)} f(x) d\sigma \geq 0\end{aligned}$$

which shows the extremality of Ω^*.

5. Multiple node case

The approach illustrated in the previous sections can be applied to the construction of extended cubature formulae that are based on integrals of f and its consecutive normal derivatives (or other differential operators of f) over fixed hyperspheres $S(R_1), \dots, S(R_k)$. To do so we need to follow, by analogy, the so called multiple node quadrature formulae in the univariate case. These are formulae of the form

$$\int_{-1}^{1} \mu(t) f(t)\, dt \approx \sum_{i=1}^{m} \sum_{\lambda=0}^{\nu_i - 1} a_{i\lambda} f^{(\lambda)}(t_i). \tag{5.1}$$

Given any set of distinct nodes $t_1 < \cdots < t_m$ in $[-1, 1]$ and the corresponding multiplicities $\nu_1, \dots, \nu_m$ ($M := \nu_1 + \cdots + \nu_m - 1$) there is a unique quadrature formula of the form (5.1) that integrates exactly all algebraic polynomials of degree M. It can be obtained by integration of the Hermite interpolating polynomial

$$H_M(f;t) = \sum_{i=1}^{m} \sum_{\lambda=0}^{\nu_i-1} f^{(\lambda)}(t_i)\Phi_{i\lambda}(t)$$

where the basic polynomials $\{\Phi_{i\lambda}\}$ are determined by the the conditions

$$\Phi_{i\lambda} \in \pi_{M-1}, \qquad \Phi_{i\lambda}^{(l)}(t_j) = \delta_{ij}\delta_{l\lambda}, \qquad i,j = 1,\dots,m, \;\; l,\lambda = 0,\dots,\nu_i - 1.$$

The Gaussian approach to multiple node quadrature formulae has encountered serious difficulties that arise from the non-linearity of the corresponding extremal problems associated with the characterisation of the extremal nodes. It was easy to observe that for given $\nu_1,\dots,\nu_m$ the highest algebraic degree of precision that can be achieved by a quadrature formula of the form (5.1) is $\hat{\nu}_1 + \cdots \hat{\nu}_m - 1$, where $\hat{\nu}_i$ denotes the smallest even number which is greater than or equal to ν_i. For example, if ν_i is odd then $\hat{\nu}_i = \nu_i + 1$. The complete characterisation of the quadrature formulae of type (5.1) that have a highest algebraic degree of precision has been done by the efforts of several outstanding mathematicians. Firstly Turán [10] proved the existence and uniqueness, and gave the characterisation of the optimal nodes of (5.1) in the particular case when all multiplicities are equal, i.e., in the case $\nu_1 = \cdots = \nu_m = \nu$. Then the Bulgarian mathematician L. Tschakaloff [9] proved the existence of optimal nodes for any fixed system of multiplicities $\{\nu_i\}$. The uniqueness stayed as an open problem for more than 20 years. In 1975 Ghizzetti and Ossicini [5] published an elegant and ingenious proof of the uniqueness of the optimal nodes. Independently the uniqueness was shown in a more general situation by Karlin and Pinkus [7].

We are going to show here that the quadratures (5.1) have natural analogues in the polyharmonic setting. In order to do this we first need to present the following polyharmonic extension of the Hermite representation of algebraic polynomials.

Corollary 4. *Let $0 < t_1 < \cdots < t_m \leq 1$ be fixed numbers and $\nu_1,\dots\nu_m$ be any given multiplicities associated with them, such that $M = \nu_1 + \cdots + \nu_m - 1$. Set $R_i^2 := t_i$, $i = 1,\dots,m$. Then every M-harmonic function u can be presented in the form*

$$u(x) = \sum_{i=1}^{m} \sum_{j=0}^{\nu_i-1} \Phi_{ij}(|x|^2)h_{ij}(x) \tag{5.2}$$

where h_{ij} are harmonic functions on B and Φ_{ij} are associated with $t_1,\dots,t_m$.

Moreover, h_{ij} is a harmonic function on B which is uniquely determined by the values of $\frac{\partial^j}{\partial\nu^j}u(x)$, on $S_i := S(R_i^2)$, for $j = 0,\dots,\nu_i - 1$ $(i = 1,\dots,m)$.

Proof. The existence of the harmonic coefficients follows immediately from Lemma 2 taking into account that the polynomials $\{\Phi_{ij}\}$ constitute a basis in π_{M-1}.

To determine the functions h_{qj}, notice that for $\lambda \in \{0,\dots,\nu_k - 1\}$

$$\left.\frac{\partial^\lambda}{\partial\nu^\lambda}u(x)\right|_{S_k} = \sum_{i=1}^{m}\sum_{j=0}^{\nu_i-1}\left.\left\{\frac{\partial^\lambda}{\partial\nu^\lambda}\left(\Phi_{ij}(|x|^2)h_{ij}(x)\right)\right\}\right|_{S_k}$$

$$= \sum_{i=1}^{m}\sum_{j=0}^{\nu_i-1}\sum_{s=0}^{\lambda}\binom{\lambda}{s}\left[\Phi_{ij}(t^2)\right]^{(s)}\Big|_{t=R_k}\left\{\frac{\partial^{(\lambda-s)}}{\partial\nu^{(\lambda-s)}}h_{ij}(x)\right\}\Bigg|_{S_k}$$
$$= \sum_{j=0}^{\nu_k-1}\sum_{s=0}^{\lambda}\binom{\lambda}{s}\left[\Phi_{kj}(t^2)\right]^{(s)}\Big|_{t=R_k}\left\{\frac{\partial^{(\lambda-s)}}{\partial\nu^{(\lambda-s)}}h_{kj}(x)\right\}\Bigg|_{S_k}.$$

The last equality is justified by the observation that

$$[\Phi_{ij}(t^2)]^{(s)}\Big|_{t=R_k} = 0 \quad \text{for } i \neq k, \quad s = 0, 1, \dots, \nu_k - 1,$$

as seen from the definition of Φ_{ij}. Besides,

$$[\Phi_{kj}(t^2)]^{(s)}\Big|_{t=R_k} = 0 \quad \text{for } s < j$$

and

$$[\Phi_{kj}(t^2)]^{(j)}\Big|_{t=R_k} = (2R_k)^j + \Phi_{kj}(R_k^2)^{(j)} + \{\cdots\}\Big|_{t=R_k},$$

where the expression in $\{\cdots\}$ is a linear combination of derivatives of Φ_{ij} of order smaller than j. Then it vanishes at $t = R_k$. Recalling that $\Phi_{kj}(R_k^2)^{(j)} = 1$ by definition we finally get

(5.3)
$$\frac{\partial^\lambda}{\partial\nu^\lambda}u(x)\Bigg|_{S_k} = \sum_{j=0}^{\lambda-1}\sum_{s=0}^{\lambda}\binom{\lambda}{s}\left[\Phi_{kj}(t^2)\right]^{(s)}\Big|_{t=R_k}\frac{\partial^{(\lambda-s)}}{\partial\nu^{(\lambda-s)}}h_{kj}(x)\Bigg|_{S_k} + (2R_k)^\lambda h_{k\lambda}(x)\Big|_{S_k}.$$

Thus, if $h_{k0}, \dots, h_{k,\lambda-1}$ are found, we can determine $h_{k\lambda}$ uniquely from this relation. The proof is complete. ∎

Now we are going to integrate (5.2) over the ball in order to get a cubature formula. The following observation will be very useful.

LEMMA 4. *If h is a harmonic function on B, then*

$$\int_S \frac{\partial^k h(\xi)}{\partial\nu^k}\, d\sigma(\xi) = 0 \quad \textit{for every } k = 1, 2, \dots$$

Proof. The claim is well-known for $k = 1$. It follows directly from the Green formula

$$\int_B \Delta h\, dx = \int_S \frac{\partial}{\partial\nu}h(\xi)\; d\sigma(\xi).$$

Then one can prove the lemma by induction. To do this, note first the fact that if h is harmonic in B, then the function

$$\sum_1^n x_i \frac{\partial h(x)}{\partial x_i}$$

is harmonic too (see Proposition 1.1 in [2]). Since

$$\frac{\partial h(x)}{\partial\nu} = \sum_1^n \frac{x_i}{|x|}\frac{\partial h(x)}{\partial x_i}$$

at any point $x \in S$, we conclude that the function $g_1(x) := |x|\frac{\partial h}{\partial\nu}$, and consequently the k-th iterate

$$g_k(x) := |x|\frac{\partial g_{k-1}}{\partial\nu}, \quad k = 1, 2, \dots \quad (g_0 := h)$$

is harmonic. Next we show that g_k can be presented in the form

$$g_k(x) = |x|^k \frac{\partial^k h}{\partial \nu^k} + \sum_{j=1}^{k-1} \varphi_{kj}(|x|) \frac{\partial^j h}{\partial \nu^j} \tag{5.4}$$

with certain functions $\varphi_{kj}(t)$. Indeed, assume that the representation holds for g_k. Then, taking into account that $\frac{\partial |x|}{\partial \nu} = 1$, we get

$$\begin{aligned} g_{k+1} = |x| \frac{\partial g_k}{\partial \nu} &= |x| \left\{ |x|^k \frac{\partial^{k+1} h}{\partial \nu^{k+1}} + k|x|^{k-1} \frac{\partial^k h}{\partial \nu^k} \right. \\ &+ \left. \sum_{j=1}^{k-1} [\varphi_j'(x) \frac{\partial^j h}{\partial \nu^j} + \varphi_j(|x|) \frac{\partial^{j+1}}{\partial \nu^{j+1}}] \right\} \\ &= |x|^{k+1} \frac{\partial^{k+1} h}{\partial \nu^{k+1}} + \sum_{j=1}^{k} \varphi_{k+1,j}(|x|) \frac{\partial^j h}{\partial \nu^j} \end{aligned}$$

where the functions $\varphi_{k+1,j}$ can be given explicitly in terms of φ_{kj}. Having (5.4) proven we use it to perform the induction step. Assume that

$$\int_S \frac{\partial^j h}{\partial \nu^j} \, d\sigma = 0 \quad \text{for} \quad j = 1, \ldots, k-1,$$

and for every harmonic function h. Since g_{k-1} is harmonic, then

$$\begin{aligned} 0 &= \int_S \frac{\partial g_{k-1}}{\partial \nu} \, d\sigma = \int_S |x| \frac{\partial g_{k-1}}{\partial \nu} \, d\sigma = \int_S g_k(x) \, d\sigma \\ &= \int_S |x|^k \frac{\partial^k h}{\partial \nu^k} \, d\sigma + \sum_{j=1}^{k-1} \varphi_{kj}(|x|) \int_S \frac{\partial^j h}{\partial \nu^j} \, d\sigma \\ &= \int_S \frac{\partial^k h}{\partial \nu^k} \, d\sigma. \end{aligned}$$

The induction is completed and so is the proof of the lemma. □

Now integrating over B both sides of the Hermite representation (5.2) and making use of the radiality of some of the terms we get as in the Lagrangean case the following extended cubature formula for polyharmonic functions.

THEOREM 4. *Let $0 < t_1 < \cdots < t_m \leq 1$ be fixed numbers and $\nu_1, \ldots \nu_m$ be any given multiplicities associated with them, such that $M = \nu_1 + \cdots + \nu_m$. Then there exists a unique cubature formula of the form*

$$\int_B f(x) \mu(|x|) \, dx \approx \sum_{k=1}^{m} \sum_{\lambda=0}^{\nu_k - 1} A_{k\lambda} \int_{S_k} \frac{\partial^\lambda}{\partial \nu^\lambda} u(\xi) \, d\sigma(\xi) \tag{5.5}$$

which integrates exactly all M-harmonic functions. Moreover,

$$A_{k,\nu_k-1} = \frac{1}{2^{\nu_k - 1} R_k^{2(n-1)+\nu_k-1}} \int_0^1 \mu(t) \, \Phi_{k,\nu_k-1}(t^2) t^{n-1} \, dt.$$

Proof. Integrating (5.2) we get

$$\int_B \mu(|x|) \, u(x) \, dx = \sum_{k=1}^{m} \sum_{\lambda=0}^{\nu_k - 1} \int_0^1 \mu(t) \, \Phi_{k\lambda}(t^2) \int_{S(t)} h_{k\lambda}(\xi) \, d\sigma(\xi) \, dt. \tag{5.6}$$

Note that

$$\int_{S(t)} h_{k\lambda}(\xi)\ d\sigma(\xi) = \gamma_n t^{n-1} h_{k\lambda}(0) = \left(\frac{t}{R_k^2}\right)^{n-1} \int_{S_k} h_{k\lambda}(\xi)\ d\sigma(\xi).$$

Now integrating (5.3) and using Lemma 4 we find

$$\begin{aligned}\int_{S_k} \frac{\partial^\lambda}{\partial\nu^\lambda} u(\xi)\ d\sigma(\xi) = \quad & \int_{S_k} \sum_{j=0}^{\lambda-1} [\Phi_{kj}]^{(\lambda)}\Big|_{t=R_k} \int_{S_k} h_{kj}(\xi)\ d\sigma(\xi) \\ & +(2R_k)^\lambda \int_{S_k} h_{k\lambda}(\xi)\ d\sigma(\xi).\end{aligned}$$

This is a linear system of equations in unknowns

$$X_\lambda := \int_{S_k} h_{k\lambda}(\xi)\ d\sigma(\xi), \quad \lambda = 0, \dots, \nu_k - 1.$$

The determinant is triangular with non-zero entries in the diagonal, thus non-zero. Then the quantities X_λ are uniquely determined as linear combinations of

$$Y_j := \int_{S_k} \frac{\partial^j}{\partial\nu^j} u(\xi)\ d\sigma(\xi), \quad j = 0, \dots, \lambda.$$

More precisely,

$$X_\lambda = \frac{1}{(2R_k)^\lambda} Y_\lambda + \sum_{j=0}^{\lambda-1} \alpha_{kj} Y_j$$

with certain coefficients $\{\alpha_{kj}\}$. Then (5.6) becomes

$$\int_B \mu(|x|) u(x)\, dx = \sum_{k=1}^{m} \sum_{\lambda=0}^{\nu_k-1} \int_0^1 \mu(t)\, \Phi_{k\lambda}(t^2) \left(\frac{t}{R_k^2}\right)^{n-1} dt \{\frac{1}{(2R_k)^\lambda} Y_\lambda + \sum_{j=0}^{\lambda-1} \alpha_{kj} Y_j\}$$

which is a formula of the desired form (5.5). It is clear that the coefficient A_{k,ν_k-1} can be found by the formula given in the theorem. The existence part of the theorem is proved.

In order to show the uniqueness, put $u(x) := \Phi_{kj}(|x|^2)$ in (5.5). Then we get

$$\int_B \mu(|x|)\, \Phi_{kj}(|x|^2)\, dx = \sum_{\lambda=0}^{\nu_k-1} A_{k\lambda} \gamma_n R_k^{n-1} \left.\{\Phi_{kj}(t^2)\}^{(\lambda)}\right|_{t=R_k}, \quad j = 0, \dots, \nu_k - 1,$$

which is a linear system for $\{A_{k,\nu_k-1}\}$ with a non-zero determinant. Thus the coefficients of the cubature are determined uniquely. $\square$

Let us turn now to the optimal choice of the nodes. Assume that $\nu_1, \dots \nu_m$ are arbitrary positive integers. Set $M := \nu_1 + \cdots + \nu_m$ and consider cubatures of the form (5.5)

Using the example

$$f(x) = (|x|^2 - t_1)^{\nu_1+1} \cdots (|x|^2 - t_m)^{\nu_m+1}$$

one can deduce that the maximal $PHOP(5.5) \le M + m$. The next theorem characterises the optimal cubature of this type. Set

$$\Omega(t^2) := (t^2 - t_1)^{\nu_1-1} \cdots (t^2 - t_m)^{\nu_m-1}.$$

THEOREM 5. *For any given set of odd multiplicities* $\nu_1, \ldots, \nu_m$ *with* $M = \nu_1 + \cdots + \nu_m$ *there exists a unique extended cubature formula of the form* (5.5) *that is exact for all* $(M+m)$*-harmonic functions. The nodes* $R_1, \ldots R_m$ *of this cubature are located at the zeros of the polynomial* $\Omega(t)$ *satisfying the orthogonality relations:*

$$\int_{-1}^{1} \Omega(t^2) Q(t^2) t^{n-1} \mu(t^2)\, dt = 0 \quad \text{for every} \quad Q \in \pi_{m-1}.$$

Proof. Consider the interpolatory type cubature with multiplicities $\{\nu_k + 1\}$. It would produce a rule of a maximal polyharmonic order of precision if $A_{k,\nu_k} = 0$. But, as seen from Theorem 4,

$$A_{k,\nu_k} = C \int_0^1 \prod_{s=1}^{m} (t - t_s)^{\nu_s} Q_k(t) \mu(t)\, dt$$

where Q_k is a polynomial of degree $m-1$ and C is some constant. Thus the existence and uniqueness of the extremal points t_k follows from the corresponding univariate results. □

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