ASYMPTOTICS OF ZEROS OF POLYNOMIALS ARISING FROM RATIONAL INTEGRALS

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ABSTRACT. We prove that the zeros of the polynomials $P_m(a)$ of degree $m$, defined by Borosh and Moll via

$$P_m(a) = \frac{2^{m+3/2}}{\pi} (a + 1)^{m+1/2} \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}},$$

approach the lemniscate $\{\zeta \in \mathbb{C} : |\zeta^2 - 1| = 1, \Re \zeta < 0\}$, as $m$ diverges.

1. INTRODUCTION

Recently Moll, Borosh and their coauthors [1, 2, 3, 6] investigated thoroughly the rational integral

$$N(a, m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}},$$

its dependence both on the parameters $m$ and $a$, and established various interesting properties of $N(a, m)$. We refer to a recent personal account of Moll [6] about the motivation for the study of the integrals (1) and about some unexpected interplays. Among the other facts, they proved that, for every fixed $m \in \mathbb{N}$,

$$P_m(a) = \frac{2^{m+3/2}}{\pi} (a + 1)^{m+1/2} N(a, m)$$

is a polynomial in $a$ of degree $m$. It was observed in [1] that the zeros of $P_m(a)$ possess a pretty regular asymptotic behaviour and nice pictures in support of this observation were furnished in [1, 6]. It was pointed out by Borosh and Moll in [1], that their numerical experiments suggest that the zeros go to a lemniscate but they were not able to predict its equation. We prove that the limit curve for the zeros of $P_m(a)$ is the left half of the lemniscate of Bernoulli

$$L = \{\zeta \in \mathbb{C} : |\zeta^2 - 1| = 1, \Re \zeta < 0\}.$$

The polar equation of $L$ is

$$\rho^2 = 2 \cos 2\theta, \quad \theta \in (3\pi/4, 5\pi/4).$$

We also try to explain the lack of zeros around the part of $L$ which is close to the origin, a fact which can be observed in the above mentioned pictures and in the Figure 1 below which shows the zeros of $P_{70}(a)$.

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Theorem 1. If $Z_m$ is the set of the zeros of $P_m(a)$, then
\[
\max_{a \in Z_m} \min \{|a - \zeta| : \zeta \in \mathbb{C}, |\zeta^2 - 1| = 1, \Re \zeta < 0\} \to 0 \text{ as } n \to \infty.
\]
Moreover, if $m \in \mathbb{N}$ is fixed, then
\[
|a + 1| \leq 1 - \frac{2 \left( \sqrt{m(m+1)(4m+1)} - m \right)}{(2m+1)^2}
\]
for every $a \in Z_m$.

The zeros of $P_{70}(z)$, together with the lemniscate $L$ and the circumference
\[
C_m = \left\{ \zeta \in \mathbb{C} : |\zeta + 1| = 1 - \frac{2 \left( \sqrt{m(m+1)(4m+1)} - m \right)}{(2m+1)^2} \right\},
\]
for $m = 70$, can be seen in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The zeros of $P_{70}(a)$, the lemniscate $L$ and the circumference $C_{70}$.}
\end{figure}

2. Proof of the theorem

We begin with a simple technical lemma which shows that the inverse polynomial of a hypergeometric polynomial is also hypergeometric. Recall that the hypergeometric function is defined by
\[
F(\alpha, \beta; \gamma; x) := _2F_1(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!},
\]
where $(\alpha)_k$ denotes the Pochhammer symbol, $(\alpha)_k := \alpha (\alpha + 1) \cdots (\alpha + k - 1)$, for $k \in \mathbb{N}$, and $(\alpha)_0 := 1$.

Lemma 1. If $n \in \mathbb{N}$, $\beta, \gamma \in \mathbb{C}$, with $\beta, \gamma \neq 0, -1, -2, \ldots, -n$, then
\[
F(-n, \beta; \gamma; z) = \frac{(\beta)_n}{(\gamma)_n} (-z)^n F(-n, 1 - \gamma - n; 1 - \beta - n; 1/z).
\]
Proof. Calculating the coefficient $d_k$ of $z^k$ on the right-hand side of (2), we obtain
\[
d_k = (-1)^n \frac{n!}{(\gamma)_n} \frac{(-n)_{n-k}(1 - \gamma - n)_{n-k}}{(1 - \beta - n)_{n-k}} \frac{1}{(n-k)!}
\]
\[
= \frac{(-1)^k n!}{(\gamma)_k} n(n-1) \cdots (n-k+1)
\]
\[
= \frac{(\beta)_k}{(\gamma)_k} (-n)_k \frac{1}{k!}
\]
and the latter coincides with the coefficient of $z^k$ on the left-hand side of (2). □

We shall need the following version of the classical Eneström-Kakeya theorem (see Exercise 2 on p. 137 in [5]):

**Theorem A.** All the zeros of the polynomial $f(z) = c_0 + c_1 z + \cdots + c_n z^n$, having positive coefficients $c_j$, lie in the ring
\[
\min_{0 \leq k \leq n-1} \left( \frac{c_k}{c_{k+1}} \right) \leq |z| \leq \max_{0 \leq k \leq n-1} \left( \frac{c_k}{c_{k+1}} \right).
\]

**Proof of Theorem 1.** It was proved in [2, 3] that $P_m(a)$ is a hypergeometric polynomial,
\[
P_m(a) = \binom{2m}{m} \text{F}(-m, m + 1; 1/2 - m; (a+1)/2).
\]
On using this representation and applying identity (2) for $n = m$, $\beta = m + 1$, $\gamma = 1/2 - m$ and $z = (a+1)/2$, we obtain
\[
P_m(a) = (-1)^m \binom{2m}{m} \binom{m + 1}{1/2 - m} \left( \frac{a+1}{2} \right)^m \text{F}(-m, 1/2; -2m; \frac{2}{a+1}).
\]
On the other hand, Driver and Möller [4] proved that, for any real positive $b$, the zeros of the polynomials
\[
w^n \text{F}(-n, b; -2n; 1/w)
\]
approach the Cassini curve
\[
|(2w - 1)^2 - 1| = 1
\]
as $n$ diverges. Their result and the representation (3) immediately imply that the zeros of $P_m(a)$ tend to the lemniscate $|\zeta^2 - 1| = 1$ as $m$ goes to infinity. In order to prove that these zeros remain in the disc $D_m$ surrounded by the circumference $C_m$, we apply Theorem A. The coefficients $c_k$ in the expansion
\[
\text{F}(-m, m + 1; 1/2 - m; (a+1)/2) = \sum_{k=0}^m c_k (a+1)^k
\]
are positive. Moreover
\[
\Delta(k) := \frac{c_k}{c_{k+1}} = \frac{(k+1)(2m-2k-1)}{(m-k)(m+k+1)}.
\]
Straightforward calculations show that,
\[
\Delta'(\kappa) = \frac{(2m-1)\kappa^2 - 2(2m^2 + 1)\kappa + 2m^3 - m^2 - m - 1}{(m-\kappa)^2(m+\kappa+1)^2},
\]
and $\Delta'(\kappa) = 0$ only for

$$\kappa_{1,2} = \frac{2m^2 + 1 \pm \sqrt{m(m+1)(4m+1)}}{2n-1},$$

where $0 < \kappa_1 < m-1 < \kappa_2$. It is easy to see that $\kappa_1$ is a point of local and also of global maximum of $\Delta(\kappa)$ when $\kappa$ varies in $[0, m-1]$. Then, by Theorem A we conclude that the zeros of $P_m(a)$ lie in the disc $|a + 1| \leq \Delta(\kappa_1)$, where

$$\Delta(\kappa_1) = 1 - \frac{2 \left( \sqrt{m(m+1)(4m+1)} - m \right)}{(2m+1)^2}.$$

This completes the proof of the theorem.

3. Remarks and open questions

While the proof of the convergence of the zeros of $P_m(a)$ to $L$ is a matter of simple transformation of Driver and Möller’s result, the lack of zeros close to the origin seems to be an interesting phenomenon. Except for the fact that $Z_m$ is a subset of $D_m$, one may obtain other regions which do not contain zeros of $P_m$.

An application of a generalization of Descartes’ rule of signs, due to Obrechkoff [7] (see Theorem 41.3 in [5]), implies

$$|\arg(a + 1)| \geq \pi/m \text{ for every } a \in Z_m.$$

In other words, the zeros of $P_m(a)$ are outside the infinite sector with vertex $-1$ which contains the ray $(-1, \infty)$, with angle $2\pi/m$.

Another method which is applicable to our case is the so-called Parabola theorem of Saff and Varga [8]. It guarantees that the zeros of certain polynomials that satisfy a three-term recurrence relation belong to a parabola region. It can be shown that the polynomials

$$q_k(z) = \frac{(1/2 - m)_k}{(-1/2 - 2m)_k} F(-k, m+1, 1/2 - m, 1 + z)$$

satisfy the recurrence relation

$$q_k(z) = \left( \frac{z}{b_k} + 1 \right) q_{k-1}(z) - \frac{z}{c_k} q_{k-2}(z), \quad k = 1, \ldots, m,$$

where

$$b_k = \frac{2m - k + 3/2}{m + k} \quad \text{and} \quad c_k = \frac{(2n - k + 3/2)(2n - k + 5/2)}{(k-1)(n - k + 3/2)}.$$

Then if $a \in Z_m$, with $a = \xi + i\eta$, the parabola theorem yields

$$\eta^2 > \frac{2(4m+1)}{2m-1} \left( \xi + \frac{3}{4m-2} \right).$$

Let us apply the left-hand side estimate in Theorem A to the Maclaurin expansion

$$P_m(a) = \sum_{k=0}^{m} d_k(m)a^k.$$

In [1] Borosh and Moll obtained the representation

$$d_k(m) = 2^{-2m} \sum_{j=k}^{m} 2^j \binom{2m - 2j}{m - j} \binom{m + j}{k}.$$
for the coefficients of this expansion and proved that the first half of the sequence \( \{d_k(m)\}_{k=0}^m \) is increasing and the second one is decreasing. Numerical experiments show that, for every fixed \( m \), the minimum of \( d_k(m)/d_{k+1}(m) \) is attained for \( k = 0 \) and the values of \( d_0(m)/d_1(m) \) behave like \( 1/m \) as \( m \) diverges.

Summarizing, these application of the results of Obrechkoff, of Saff and Varga and the Eneström-Kakeya theorem guarantee the lack of zeros in very “tiny” regions around the origin. More precisely, the circumference \( C_m \) cuts a larger portion of the lemniscate \( L \) around the origin in comparison with the cuts with the above mentioned sector, parabola and circumference of radius \( 1/m \) around the origin. The author finds that a promising way of determining the precise behaviour of the closest to the origin zero of \( P_m(a) \) is the relation between \( Z_m \) and the zeros of the polynomials \( H_n(b, u) \), defined by (4.4) in [4]. There, on p. 86, Driver and Möller formulated a challenging conjecture about the location of the zeros of \( H_n(b, u) \). If true, and once it is established, the statement of that conjecture will provide precise information about the location of the zeros of \( P_m(a) \).

We finish with an observation motivated by another result of Driver and Möller. Proposition 5.2 in [4] states that the zeros of the polynomials (4) lie outside the Cassini curve (5) for all \( n \in \mathbb{N} \), provided \( b \geq 1 \). In our case \( b = 1/2 \), so we can not conclude that the zeros of \( P_m(a) \) lie outside \( L \) though numerical experiments show that they do.

References


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