# DIFFERENTIAL OPERATORS OF INFINITE ORDER AND THE DISTRIBUTION OF ZEROS OF ENTIRE FUNCTIONS

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*Dedicated to the memory of Professor R. P. Boas, Jr.*

**1. Introduction and notation.** A real entire function $\phi(x)$ is said to be in the *Laguerre-Pólya class* if $\phi(x)$ can be expressed in the form

$$\phi(x) = cx^n e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} \left(1+\frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

where $c, \beta, x_k \in \mathbb{R}$, $\alpha \geq 0$, $n$ is a nonnegative integer and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. If $\phi(x)$ is in the Laguerre-Pólya class, we shall write $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}$. A function $\phi(x)$ is said to be in $\mathcal{L}\text{-}\mathcal{P}^*$ if $\phi(x)$ can be expressed in the form $\phi(x) = p(x)\psi(x)$, where $p(x)$ is a real polynomial and $\psi(x) \in \mathcal{L}\text{-}\mathcal{P}$. In particular, functions in $\mathcal{L}\text{-}\mathcal{P}^*$ have only finitely many nonreal zeros.

Let $D := \dfrac{d}{dx}$ denote differentiation with respect to $x$. If

$$\phi(x) = \sum_{k=0}^{\infty} \alpha_k x^k \qquad (\alpha_k \in \mathbb{R}) \tag{1.1}$$

is a formal power series, we define the operator $\phi(D)$ by

$$\phi(D)f(x) = \sum_{k=0}^{\infty} \alpha_k f^{(k)}(x), \tag{1.2}$$

whenever the right-hand side of (1.2) represents an analytic function in a neighborhood of the origin. When $\phi(x)$ is an entire function, the operator $\phi(D)$ has been studied by several authors (see, for example, [5, §11], [19, Chapter IX], [22] and [32]).

The conjecture of Pólya and Wiman, proved in [8], [9] and [17], states that if $f(x) \in \mathcal{L}\text{-}\mathcal{P}^*$, then $D^m f(x)$ is in the Laguerre-Pólya class for all sufficiently large positive integers $m$. In this paper, we analyze the more general situation when $D$ is replaced by the operator $\phi(D)$. In Section 2, we consider real power series with zero linear term (i.e., $\alpha_1 = 0$ in (1.1)) and $\alpha_0\alpha_2 < 0$, and show that if $f(x)$ is any real polynomial, then $[\phi(D)]^m f(x) \in \mathcal{L}\text{-}\mathcal{P}$ for all sufficiently large positive integers $m$ (Theorem 2.4). If the linear term in (1.1) is nonzero,

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then simple examples show (cf. Section 3) that this fails without much stronger restrictions on $\phi$. In this case, we show that if $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}$ and $f(x) \in \mathcal{L}\text{-}\mathcal{P}^*$ (with some restriction of the growth of $\phi$ or $f$) and if $\phi(x)$ has at least one real zero, then $[\phi(D)]^m f(x) \in \mathcal{L}\text{-}\mathcal{P}$ for all sufficiently large positive integers $m$ (Theorem 3.3). A separate analysis of the operator $e^{-\alpha D^2}$, $\alpha > 0$ (cf. Theorem 3.10, Corollary 3.11), shows that, not only does the previous result hold, but that the zeros become simple. In fact, if $f(x) \in \mathcal{L}\text{-}\mathcal{P}$ with order less than two, then $e^{-\alpha D^2} f(x)$ has only real *simple* zeros. The question of simplicity of zeros is pursued further in Section 4, where we generalize a result of Pólya [26]. We prove that if $\phi(x)$ and $f(x)$ are functions in the Laguerre-Pólya class of order less than two and $\phi$ has an infinite number of zeros, and if there is a bound on the multiplicities of the zeros of $f$, then $\phi(D)f(x)$ has only simple real zeros (Theorem 4.6).

We next mention some terminology which will be used throughout this paper. If $\phi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k!$ is a real entire function, then the $n$-th *Jensen polynomial* associated with $\phi(x)$ is defined by $g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$, $n = 0, 1, 2, \ldots$. Moreover, for any polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, $a_n \neq 0$, we define $p^*(x) = x^n p(1/x) = \sum_{k=0}^{n} a_{n-k} x^k$. In particular, we have $\phi(D)x^n = g_n^*(x)$ for each $n$, $n = 0, 1, 2, \ldots$. The following proposition summarizes two basic properties of $\phi(D)$.

**Proposition 1.1.** *Let* $\phi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k! \in \mathcal{L}\text{-}\mathcal{P}$. *Let* $[\phi(x)]^m = \sum_{k=0}^{\infty} \gamma_{k,m} x^k/k!$ *(*$m = 1, 2, 3, \ldots$*), where* $\gamma_{k,1} = \gamma_k$ *for each* $k$. *Let* $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$ *be a polynomial of degree* $n$. *Then*

(1) $g_{k,m}^*(x) = [\phi(D)]^m x^k \in \mathcal{L}\text{-}\mathcal{P} \qquad (m = 1, 2, 3, \ldots; k = 1, 2, 3, \ldots)$.
(2) $[\phi(D)]^m f(x) = \sum_{k=0}^{n} a_k g_{k,m}^*(x)$.

*Proof.* Since $\phi \in \mathcal{L}\text{-}\mathcal{P}$, we also have $\phi^m \in \mathcal{L}\text{-}\mathcal{P}$ and hence (1) follows from the Hermite-Poulain Theorem [24, p. 4]. Since $[\phi(D)]^m$ is a linear operator, part (2) is clear. □

We conclude this introduction by citing a few selected items from the extensive literature dealing with the differential operator $\phi(D)$, where $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}$. In connection with the study of the distribution of zeros of certain Fourier transforms, Pólya [27] characterized the universal factors in terms of $\phi(D)$, where $\phi \in \mathcal{L}\text{-}\mathcal{P}$. Subsequently, this work of Pólya was extended by de Bruijn [5], who studied, in particular, the operators $\cos(\lambda D)$ and $e^{-\lambda D^2}$, $\lambda > 0$. Benz [2] applied the operator $1/\phi(D)$, $\phi \in \mathcal{L}\text{-}\mathcal{P}$, to investigate the distribution of zeros of certain exponential polynomials. The operators $\phi(D)$, $\phi \in \mathcal{L}\text{-}\mathcal{P}$, play a central role in Schoenberg's celebrated work [31] on Pólya frequency functions and totally positive functions. Hirschman and Widder [16] used $\phi(D)$, $\phi \in \mathcal{L}\text{-}\mathcal{P}$, to develop the inversion and representation theories of certain convolution transforms. More recently, Boas and Prather [4] considered the final set problem for certain trigonometric polynomials when differentiation $D$ is replaced by $\phi(D)$.

**2. Power series with zero linear term.** We begin with some lemmas which help us to establish the limiting behavior of $[\phi(D)]^m f(x)$ as $m$ tends to infinity.

**Lemma 2.1.** *Let*

$$h(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k \qquad (\alpha_k \in \mathbb{R})$$

*be any power series and let*

$$h_m(x) = [h(x)]^m = \sum_{k=0}^{\infty} \frac{\beta_k}{k!} x^k \qquad (\beta_k = \beta_k(m), \beta_0 = 1; m = 1, 2, 3, \ldots).$$

*Then*

$$\beta_n = \frac{1}{n} \sum_{j=1}^{n} \binom{n}{j} [j(m+1) - n] \alpha_j \beta_{n-j} \qquad (n = 1, 2, 3, \ldots).$$

*Proof.* This is a familiar formula for the power of a power series (cf. [14, p. 14]). □

We shall make use of the standard notations $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ and $(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n)$. Following usual practice, we use the convention $\binom{k}{i} = 0$ if $i > k$.

**Lemma 2.2.** *Let $h(x) = \sum_{k=0}^{\infty} \alpha_k x^k / k!$, where $\alpha_0 = 1$ and $\alpha_1 = 0$, be a real power series. For each $m = 1, 2, 3, \ldots$, let $h_m(x) = [h(x)]^m = \sum_{k=0}^{\infty} \beta_k(m) x^k / k!$, so that $h_m(x)$ is also a power series with $\beta_0 = 1$ and $\beta_1 = 0$. Then*

$$\lim_{m \to \infty} \frac{\beta_{2n}(m)}{m^n} = (2n-1)!! \alpha_2^n \qquad (n = 1, 2, 3, \ldots)$$

*and*

$$\lim_{m \to \infty} \frac{\beta_{2n+1}(m)}{m^{n+\frac{1}{2}}} = 0 \qquad (n = 0, 1, 2, \ldots).$$

*Proof.* We will prove the theorem by induction on $n$. By Lemma 2.1 and the induction hypothesis for $n-1$, we have

$$\begin{aligned}
\frac{\beta_{2n}(m)}{m^n} &= \frac{1}{m^n} \frac{1}{2n} \sum_{j=1}^{2n} \binom{2n}{j} [j(m+1) - 2n] \alpha_j \beta_{2n-j} \\
&= \frac{1}{m} \frac{1}{2n} \binom{2n}{2} [2(m+1) - 2n] \frac{\alpha_2 \beta_{2n-2}}{m^{n-1}} + o(1) \qquad (m \to \infty) \\
&= \frac{1}{m} [(2n-1)(m+1-n)] \alpha_2 ((2n-3)!! \alpha_2^{n-1} + o(1)) + o(1) \qquad (m \to \infty) \\
&= (2n-1)!! \alpha_2^n + o(1) \qquad (m \to \infty).
\end{aligned}$$

For the second limit,

$$\begin{aligned}\frac{\beta_{2n+1}(m)}{m^{n+\frac{1}{2}}} &= \frac{1}{m^{n+\frac{1}{2}}}\frac{1}{2n+1}\sum_{j=1}^{2n+1}\binom{2n+1}{j}[j(m+1)-(2n+1)]\alpha_j\beta_{2n+1-j} \\ &= \frac{1}{m}\frac{1}{2n+1}\binom{2n+1}{2}[2(m+1)-(2n+1)]\frac{\alpha_2\beta_{2n-1}}{m^{n-\frac{1}{2}}}+o(1) \qquad (m\to\infty) \\ &= o(1) \qquad (m\to\infty).\end{aligned}$$

The limits follow. □

**Lemma 2.3.** *Fix $\alpha > 0$. For $k = 1, 2, 3, \ldots,$ the zeros of the polynomial*

$$q_k(t;\alpha) = t^k + \sum_{j=1}^{k}\binom{k}{2j}(2j-1)!!(-1)^j\alpha^j t^{k-2j}$$

*are all real and simple.*

*Proof.* Using $(2j-1)!! = (2j)!/(2^j j!)$, we obtain

$$\begin{aligned} q_k(t;\alpha) &= \sum_{j=0}^{k}\binom{k}{2j}\frac{(2j)!}{j!}\frac{(-\alpha)^j}{2^j}t^{k-2j} \\ &= \sum_{j=0}^{[k/2]}\frac{k!}{(k-2j)!j!}\frac{(-\alpha)^j}{2^j}t^{k-2j} \\ &= \left(\frac{\alpha}{2}\right)^{k/2}H_k\left(\frac{t}{\sqrt{2\alpha}}\right),\end{aligned}$$

where $[x]$ denotes the greatest integer less than or equal to $x$ and $H_k(t)$ is the $k$-th Hermite polynomial defined by $H_k(t) = (-1)^k e^{t^2}D^k e^{-t^2}$ (cf. [30, p. 189]). Since it is known that $H_k(t)$ has only real simple zeros [30, p. 193], the assertion of the lemma follows. □

**Theorem 2.4.** *Let*

$$\phi(x) = \sum_{k=0}^{\infty}\alpha_k\frac{x^k}{k!} \tag{2.1}$$

*be a real power series with $\alpha_0 = 1$, $\alpha_1 = 0$ and $\alpha_2 < 0$. Let $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$ be a polynomial of degree at least one. Then there is a positive integer $m_0$ such that $[\phi(D)]^m f(x) \in \mathcal{L}$-$\mathcal{P}$ for all $m \geq m_0$. In fact, $m_0$ can be chosen so that all the zeros are simple.*

*Proof.* Let $G_m(x) = [\phi(D)]^m f(x)$. We can write $G_m(x) = \sum_{k=0}^{n} a_k g^*_{k,m}(x)$, where the polynomial $g^*_{k,m}(x) = x^k g_{k,m}(1/x)$ with $g_{k,m}(x) = [\phi(D)]^m x^k$. (If $[\phi(x)]^m$ is an entire function in $\mathcal{L}$-$\mathcal{P}$, then $g^*_{k,m}(x)$ is $k!$ times the $k$-th Appell polynomial associated with $[\phi(x)]^m$

[7, p. 243].) Write $[\phi(x)]^m = \sum_{k=0}^{\infty} \beta_k(m)x^k/k!$, noting that $[\phi(x)]^m$ is again a power series with $\beta_0 = 1$ and $\beta_1 = 0$. A calculation shows that $g^*_{k,m}(x) = \sum_{j=0}^{k} \binom{k}{j}\beta_j(m)x^{k-j}$. But then by Lemma 2.2,

$$\begin{aligned}\lim_{m\to\infty} \frac{g^*_{k,m}(x\sqrt{m})}{m^{k/2}} &= \lim_{m\to\infty} \sum_{j=0}^{k} \binom{k}{j} \frac{\beta_j(m)}{m^{j/2}} x^{k-j} \\ &= x^k + \sum_{j=1}^{[k/2]} \binom{k}{2j} (2j-1)!!\alpha_2^j x^{k-2j}\end{aligned}$$

Thus by Lemma 2.3, the polynomial

$$\lim_{m\to\infty} \frac{g^*_{k,m}(x\sqrt{m})}{m^{k/2}} = \left(\frac{-\alpha_2}{2}\right)^{k/2} H_k\left(\frac{x}{\sqrt{-2\alpha_2}}\right) \qquad (k \geq 1)$$

has only real simple zeros. Therefore,

$$\begin{aligned}\frac{G_m(x\sqrt{m})}{m^{n/2}} &= \sum_{k=0}^{n} a_k \frac{g^*_{k,m}(x\sqrt{m})}{m^{n/2}} \\ &= a_n \left(\frac{-\alpha_2}{2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{-2\alpha_2}}\right) + o(1) \qquad (m \to \infty)\end{aligned} \tag{2.2}$$

and consequently, it follows that for all $m$ greater than or equal to some $m_0$, $G_m(x)$ has only real simple zeros. □

While it is not particularly related to the goals of this paper, it is clear from equation (2.2) what happens if $\alpha_2 > 0$. We state the analog of Theorem 2.4 when $\alpha_2 > 0$ as a corollary.

**Corollary 2.5.** *Let $\phi(x) = \sum_{k=0}^{\infty} \alpha_k x^k/k!$ be a real power series with $\alpha_0 = 1$, $\alpha_1 = 0$ and $\alpha_2 > 0$. Let $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$ be a polynomial of degree at least one. Then there is a positive integer $m_0$ such that for all $m \geq m_0$, the zeros of the polynomial $[\phi(D)]^m f(x)$ are simple and all lie on the imaginary axis.*

**Examples.**

1. (1) If $\alpha_0 = 0$, Theorem 2.4 is trivial since $[\phi(D)]^m$ will send a polynomial to 0 when $m$ is sufficiently large.
2. (2) Theorem 2.4 does not extend, in general, to transcendental entire functions $f(x)$. For example, let $\phi(x) = 1 - x^2 + x^3$, so that $\phi$ is a polynomial with one real zero satisfying the hypotheses of Theorem 2.4. Let $f(x) = (x^2 + 2)e^x \in \mathcal{L}\text{-}\mathcal{P}^*$. Then, for $m = 1, 2, \ldots$, we obtain $[\phi(D)]^m f(x) = [x^2 + 2mx + (m+1)(m+2)]e^x$, with zeros $-m \pm i\sqrt{3m+2}$.

*Remark 2.6.* In some special cases the operators $\phi(D)$ enjoy some particularly elegant properties. For example, the operator $\sinh(\lambda D)$, $\lambda > 0$, is useful in the study of certain difference polynomials (cf. [13]). As another example, set $\phi_\lambda(D) = 2\sinh(\lambda D)/(\lambda D)$, $\lambda > 0$. Then it is not difficult to show that for any real polynomial $f(x)$,

$$\phi_\lambda(D)f(x) = \int_{x-\lambda}^{x+\lambda} f(t)\,dt \qquad (\lambda > 0).$$

Now, it is known that if the zeros of $f(t)$ lie in a vertical strip containing the origin, then this strip also contains the zeros of $\phi_\lambda(D)f(x)$ [18, p.68]. By Corollary 2.5, we know that, for all sufficiently large integers $m$, the zeros of $[\phi_\lambda(D)]^m f(x)$ lie on the imaginary axis. For this operator, it is possible to obtain more precise information about the location of zeros of the polynomials $\phi_\lambda(D)f(x)$, since in this case, we can make use of the various properties of the polynomials (cf. [30, p.214, formula (24)])

$$[\phi_\lambda(D)]^m x^k = \frac{k!}{(k+m)!}\sum_{j=0}^{m}\binom{m}{j}(-1)^j(x+(m-2j)\lambda)^{m+k},$$

$k = 0, 1, 2, \ldots$, all of whose zeros lie on the imaginary axis.

There are several open problems dealing with the distribution of zeros of certain difference polynomials. We conclude this section with the following open question of Barnard [1]. Let $p_N(x) = \prod_{k=-N}^{N}(x-k)$. Then is it true that for all $\lambda > 0$ and for all $n = 0, 1, 2, \ldots$, the zeros of the polynomial $[\sinh(\lambda D)]^n p_N(x)$ lie on the coordinate axes? When $n = 1$, this was established by Stolarsky [34]. In [1], Barnard also raises the following more general question: Characterize (all) the polynomials $f(x)$ for which all the zeros of the polynomial $[\sinh(\lambda D)]^n f(x)$ lie on the coordinate axes for all $\lambda > 0$ and $n = 0, 1, 2, \ldots$.

**3. Entire functions in the Laguerre-Pólya class.** When the linear term is present in the series (2.1) of Theorem 2.4 (i.e. $\alpha_1 \neq 0$), the limits used in the proof of Theorem 2.4 no longer apply. It turns out that the collection of power series which work must be greatly restricted (see the example below). On the other hand, we shall be able to extend Theorem 2.4 (except for the simplicity of zeros) to almost all functions $f(x)$ in $\mathcal{L}$-$\mathcal{P}^*$.

**Example.** Let $\phi(x) = \sum_{k=0}^{\infty} \alpha_k x^k/k!$ be a real power series with $\alpha_0 = 1$. When we compute $[\phi(D)]^m(x^2+bx+c)$, we see that the zeros of the resulting quadratic have discriminant equal to $b^2-4c+4m(\alpha_1^2-\alpha_2)$. Thus for large $m$, we have nonreal zeros unless $\alpha_1^2-\alpha_2 \geq 0$. This is the first of the Turán inequalities, satisfied by any $\phi(x) \in \mathcal{L}$-$\mathcal{P}$ [7]. We shall make heavy use of the special form of the functions $\phi$ in $\mathcal{L}$-$\mathcal{P}$ in establishing our results.

In the sequel, we shall be concerned with functions $\phi(x) = \sum a_k x^k$ and $f$ which, in general, are both *transcendental* entire functions. Therefore, it will be necessary for us to address the question of when the expression $\phi(D)f(x) = \sum a_k f^{(k)}(x)$ converges and represents an entire function. To this end, we shall first establish some preliminary lemmas. Lemma 3.1 is a known result (cf. [19, Theorem 8, p. 360]) which we include here for the reader's convenience as some of the inequalities in the proof will be used below. (For related results, see also [5, §11.7], [23] and [32].)

**Lemma 3.1.** *Let $\phi(x) = e^{\alpha_1 x^2}\phi_1(x)$ and $f(x) = e^{\alpha_2 x^2} f_1(x)$, where $\alpha_i \in \mathbb{R}$ for $i = 1, 2$, be entire functions where $\phi_1$ and $f_1$ have genus at most one. If $|\alpha_1\alpha_2| < 1/4$, then $\phi(D)f(x)$ is an entire function of order at most 2.*

*Proof.* Set $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$. Since $\phi$ is an entire function, not exceeding the normal type $|\alpha_1|$ of order 2, it follows that (cf. [5, Chapter 2] or [32, Lemma 5, p. 41])

$$0 \le \overline{\lim_{n\to\infty}}((n!)^{-1/2}|a_n|)^{1/n} \le \overline{\lim_{n\to\infty}}\, n^{1/2}\sqrt[n]{|a_n|} \le (2|\alpha_1|)^{1/2}, \tag{3.1}$$

where we have used the elementary inequality $n^n e^{-n} \le n!$. Next, using Cauchy's integral formula, we find that for any $R > 0$ and $n = 0, 1, 2, \ldots$, $|f^{(n)}(z)| \le \frac{n!}{R^n} M(R + |z|, f)$, where $M(r, f) = \max_{|z|=r}|f(z)|$ denotes the maximum modulus function. Now fix $\epsilon > 0$ and $r > 0$. Then, by setting $R = \sqrt{\frac{n}{2(|\alpha_2|+\epsilon)}}$ above, a calculation shows that, for all $n$ sufficiently large and for $|z| \le r$,

$$|f^{(n)}(z)| \le \frac{n!}{n^{n/2}} 2^{n/2} e^n (|\alpha_2| + \epsilon)^{n/2} e^{(|\alpha_2|+\epsilon)r^2}.$$

This inequality together with (3.1) yield, for $|z| \le r$,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n f^{(n)}(z)|} \le (4|\alpha_1\alpha_2|)^{1/2} = c < 1, \tag{3.2}$$

and consequently, $\sum_{n=0}^{\infty} |a_n f^{(n)}(z)| \le K \sum_{n=0}^{\infty} c^n < \infty$, for some constant $K$. Thus it follows that the series $\sum a_n f^{(n)}(z)$ converges uniformly on compact subsets of $\mathbb{C}$, whence $\phi(D)f$ is an entire function. Moreover, it follows that the order of $\phi(D)f$ is at most 2. □

*Remark.* The assumption that $|\alpha_1\alpha_2| < 1/4$ is necessary in the preceding lemma. To see this, we consider the following example of Pólya [25, p.243] (see also [35, p. 431] and [32, p. 106]). Let $\phi(x) = e^{-\alpha_1 x^2}$ and $f(x) = e^{-\alpha_2 x^2}$, where $\alpha_1,\ \alpha_2 > 0$. Then a calculation shows that the series $[\phi(D)f(x)]_{x=0} = \sum_{k=0}^{\infty} \dfrac{(2k)!}{(k!)^2}(\alpha_1\alpha_2)^k$ converges if and only if $0 < \alpha_1\alpha_2 < 1/4$. Indeed, $\sum_{k=0}^{\infty}\binom{2k}{k}x^k = 1/\sqrt{1-4x}$.

The Hermite-Poulain theorem, a generalization of Rolle's theorem, (cf. [24, p. 4] or [25]) does not extend to arbitrary real entire functions of order 2 having only real zeros. (Consider, for example, $f(x) = (x+1)e^{x^2/2}$; then $f'(x) = (1 + x + x^2)e^{x^2/2}$, so that $f'(x)$ does not have only real zeros.) However, the following lemma, to be used in the sequel, shows that under some additional hypotheses, the Hermite-Poulain theorem remains valid for the class $\mathcal{L}$-$\mathcal{P}^*$.

**Lemma 3.2.** *Let $\phi(x) = e^{-\alpha_1 x^2}\phi_1(x)$ and $f(x) = e^{-\alpha_2 x^2} f_1(x)$, where $\alpha_1, \alpha_2 \geq 0$, $0 \leq \alpha_1\alpha_2 < \frac{1}{4}$ and $\phi_1$ and $f_1$ are real entire functions of genus 0 or 1. If $\phi \in \mathcal{L}\text{-}\mathcal{P}$ and $f \in \mathcal{L}\text{-}\mathcal{P}^*$, then $\phi(D)f \in \mathcal{L}\text{-}\mathcal{P}^*$. Moreover, $Z_c(\phi(D)f) \leq Z_c(f)$, where $Z_c(f)$ denotes the number of nonreal zeros of $f$, counting multiplicities.*

*Proof.* We first observe that $\mathcal{L}\text{-}\mathcal{P}^*$ is closed under differentiation. Indeed, set $f(x) = p(x)\psi(x)$, where $\psi \in \mathcal{L}\text{-}\mathcal{P}$ and $p(x)$ is a real polynomial with $Z_c(p) = 2d$. Since $\psi \in \mathcal{L}\text{-}\mathcal{P}$, there exists a sequence of polynomials $\{\psi_n\}$, $\psi_n \in \mathcal{L}\text{-}\mathcal{P}$, such that $\psi_n \to \psi \quad (n \to \infty)$ uniformly on compact subsets of $\mathbb{C}$ (cf. [19, Chapter VIII]). Let $f_n(x) = p(x)\psi_n(x)$. Then $f_n(x) \to f(x)$ and $f_n'(x) \to f'(x)$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. Hence it follows from Rolle's theorem that $Z_c(f'(x)) \leq Z_c(f(x)) = 2d$. Furthermore, if we write $f'(x) = e^{-\alpha_3 x^2} f_2(x)$, where $f_2$ is a real entire function of genus 0 or 1 in $\mathcal{L}\text{-}\mathcal{P}^*$, then an argument similar to the one used by Pólya and Schur [28, p. 109] shows that $\alpha_3 \geq \alpha_2$. (In fact, it is known [15, p. 107, footnote] that $\alpha_3 = \alpha_2$.) Thus $f' \in \mathcal{L}\text{-}\mathcal{P}^*$. Since $D(e^{\gamma x} f(x)) = e^{\gamma x}(D+\gamma)f(x)$ for $\gamma \in \mathbb{R}$, the argument above also shows that for $f \in \mathcal{L}\text{-}\mathcal{P}^*$, we have $Z_c((D+\gamma)f(x)) \leq Z_c(f(x))$ and that $(D+\gamma)f \in \mathcal{L}\text{-}\mathcal{P}^*$. More generally, we have that if $q_n(x)$ is any real polynomial having only real zeros, then for any $f \in \mathcal{L}\text{-}\mathcal{P}^*$,

$$Z_c(q_n(D)f) \leq Z_c(f) \tag{3.3}$$

and $q_n(D)f \in \mathcal{L}\text{-}\mathcal{P}^*$.

Finally, to prove $\phi(D)f \in \mathcal{L}\text{-}\mathcal{P}^*$, we first note that since $0 \leq \alpha_1\alpha_2 < \frac{1}{4}$, $\phi(D)f$ is an entire function by Lemma 3.1. Set $\phi(x) = \sum_{n=0}^{\infty} a_n x^n$ and let $\phi_\nu(x) = \sum_{n=0}^{\nu} \binom{\nu}{n} n! a_n x^n$ denote the $\nu$-th Jensen polynomial associated with $\phi$. Then it is known (cf. [19, Chapter VIII] or [28]) that

$$\phi_\nu\left(\frac{x}{\nu}\right) \in \mathcal{L}\text{-}\mathcal{P} \tag{3.4}$$

for each $\nu$, and $\phi_\nu(x/\nu) \to \phi(x) \quad (\nu \to \infty)$ uniformly on compact subsets of $\mathbb{C}$. If we can prove that $\phi_\nu(D/\nu)f(x) \to \phi(D)f(x) \quad (\nu \to \infty)$ uniformly on compact subsets of $\mathbb{C}$, then it follows from (3.3), (3.4) and Hurwitz' theorem, that $\phi(D)f \in \mathcal{L}\text{-}\mathcal{P}^*$ and $Z_c(\phi(D)f) \leq Z_c(f)$. To this end, let $r > 0$ and $\epsilon > 0$ be given. Then by (3.2), there is a positive integer $m_0$ such that, for $|z| \leq r$ and some constant $K > 0$,

$$|a_n||f^{(n)}(z)| \leq Kc^n \text{ for all } n \geq m_0,\ 0 \leq c < 1. \tag{i}$$

Hence, there is a positive integer $m_1 > m_0$ such that for all $m \geq m_1$,

$$K \sum_{n=m+1}^{\infty} c^n = K\frac{c^{m+1}}{1-c} < \frac{\epsilon}{3}. \tag{ii}$$

Also, there is a positive integer $N > m_1$ such that for $|z| \leq r$ and $\nu > N$,

$$\left| \sum_{n=2}^{m_1} \left(1 - \frac{1}{\nu}\right) \cdots \left(1 - \frac{n-1}{\nu}\right) a_n f^{(n)}(z) - \sum_{n=2}^{m_1} a_n f^{(n)}(z) \right| < \frac{\epsilon}{3}. \tag{iii}$$

Therefore, for $\nu > N$ and $|z| \le r$, we have by (i), (ii) and (iii),

$$\left|\phi_\nu\left(\frac{D}{\nu}\right) f(z) - \phi(D)f(z)\right| = \Bigg|\sum_{n=2}^{m_1}\left(1-\frac{1}{\nu}\right)\cdots\left(1-\frac{n-1}{\nu}\right)a_n f^{(n)}(z) - \sum_{n=2}^{m_1} a_n f^{(n)}(z)$$
$$+ \sum_{n=m_1+1}^{\nu}\left(1-\frac{1}{\nu}\right)\cdots\left(1-\frac{n-1}{\nu}\right)a_n f^{(n)}(z) - \sum_{n=m_1+1}^{\infty} a_n f^{(n)}(z)\Bigg|$$
$$< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.$$

□

We can now prove a second part of our generalization of the Pólya-Wiman result.

**Theorem 3.3.** *Let $\phi(x) = e^{-\alpha_1 x^2}\phi_1(x)$ and $f(x) = e^{-\alpha_2 x^2} f_1(x)$, where $\alpha_1, \alpha_2 \ge 0$, $\alpha_1\alpha_2 = 0$ and $\phi_1$ and $f_1$ are real entire functions of genus 0 or 1. If $\phi \in \mathcal{L}\text{-}\mathcal{P}$, $f \in \mathcal{L}\text{-}\mathcal{P}^*$ and $\phi(x)$ has at least one real zero, then there is a positive integer $m_0$ such that for all $m \ge m_0$, we have $[\phi(D)]^m f(x) \in \mathcal{L}\text{-}\mathcal{P}$.*

*Proof.* By Lemmas 3.1 and 3.2, $[\phi(D)]^m f(x)$ is defined and lies in $\mathcal{L}\text{-}\mathcal{P}^*$ for each positive integer $m$. We can write $\phi(x) = ce^{-\alpha x^2+\beta x}x^n\prod_{k=1}^{\infty}(1-x/x_k)e^{x/x_k}$, where $c, \beta, x_k \in \mathbb{R}$, $n$ is a nonnegative integer, $\alpha_1 \ge 0$ and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. A computation shows that $e^{x_1 x}(D+x_1)^m f(x) = D^m e^{x_1 x} f(x)$. We know from the Pólya-Wiman result (cf. [8], [9], [17]) that there exists an $m$ such that $D^m e^{x_1 x} f(x)$ lies in $\mathcal{L}\text{-}\mathcal{P}$, whence so does $(D+x_1)^m f(x)$. By Lemma 3.2, we see that $[\phi(D)]^m f(x) = [e^{-\alpha D^2+\beta D}\prod_{k=2}^{\infty}(1+D/x_k)e^{-D/x_k}]^m[(D+x_1)^m f(x)]$ is also in $\mathcal{L}\text{-}\mathcal{P}$. □

We are left with the question as to what happens when $\phi$ has no zeros. If $\phi(x) = e^{\beta x}$, then $[\phi(D)]^m f(x) = f(x+m\beta)$, so the imaginary parts of the zeros do not change. On the other hand, if $\phi(x) = e^{-\alpha x^2}$, we can use very different methods to again prove our theorem. In fact, we shall obtain the stronger result that the zeros are not only real, but also simple. The simplicity cannot be true in general, as it fails for $\phi(x) = x$ when applied to entire functions with zeros of arbitrarily high multiplicity. Nevertheless, we shall see in Section 4 that simplicity of zeros is a common occurrence.

**Lemma 3.4** [5, Lemma 3]**.** *Let $\psi(z) \in \mathcal{L}\text{-}\mathcal{P}$ of order less than 2. Then there exists a sequence of polynomials $\{\psi_n(z)\}_{n=1}^{\infty}$, with only real zeros, such that*

1. *$\psi_n(z) \to \psi(z)$ uniformly on compact subsets of $\mathbb{C}$, and*
2. *for any $\eta > 0$, there exists a positive constant $C(\eta)$, independent of $n$, such that for all $z \in \mathbb{C}$,*

$$|\psi_n(z)| \le C(\eta)e^{\eta|z|^2} \text{ and } |\psi(z)| \le C(\eta)e^{\eta|z|^2}.$$

*Remark 3.5.*

(1) It is easily seen that Lemma 3.4 remains true for $\psi$ in $\mathcal{L}$-$\mathcal{P}^*$.
(2) If we set $\psi(z) = e^{\beta z} z^m \prod_{k=1}^{\infty}(1 - z/x_k)e^{z/x_k} \in \mathcal{L}$-$\mathcal{P}$, where $\beta, x_k \in \mathbb{R}$ for all $k$, $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$, and $m$ is a nonnegative integer, then the polynomials $\psi_n(z)$ in Lemma 3.4 can be defined as

$$\psi_n(z) = \left(1 + \frac{\beta z}{n}\right)^n z^m \prod_{k=1}^{n} \left(1 - \frac{z}{x_k}\right)\left(1 + \frac{z}{n x_k}\right)^n \qquad (n = 1, 2, \dots)$$

(3) The proof of our next lemma requires the following integral representation of certain functions of the form $e^{tD^2} f(x)$ [32, p. 84]. If $f$ is an entire function of order less than 2 and $0 \neq t \in \mathbb{R}$, then

$$\begin{aligned} e^{tD^2} f(x) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} f^{(2k)}(x) \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} f(x + 2s\sqrt{t})\, ds \end{aligned}$$

is also an entire function of order less than 2.

We will next combine Theorem 3.3 with the aforementioned extension of Lemma 3.4 to $\mathcal{L}$-$\mathcal{P}^*$.

**Lemma 3.6.** *Let $\psi(z)$ and $\psi_n(z)$ be defined as in the previous remark and assume that the order of $\psi$ is less than 2. Let $f(z) = p(z)\psi(z)$ and let $f_n(z) = p(z)\psi_n(z)$. Then for each fixed $t \in \mathbb{R}$, $t \neq 0$, we have $e^{tD^2} f_n(z) \to e^{tD^2} f(z)$ $(n \to \infty)$ uniformly on compact subsets of $\mathbb{C}$.*

*Proof.* Fix $t_0 \in \mathbb{R}$, $t_0 \neq 0$ and a compact subset $S \subset \mathbb{C}$. Let $r > 0$ be sufficiently large so that the closed disk $\{ z \mid |z| < r \}$ contains $S$. Let $\epsilon > 0$. Then it follows from Remark 3.5(1) that, for $0 < \eta < (16|t_0|)^{-1}$, there is a constant $C(\eta)$, independent of $n$, such that for all $w \in \mathbb{C}$, we have $|f_n(w)| \leq C(\eta)e^{\eta|w|^2}$ and $|f(w)| \leq C(\eta)e^{\eta|w|^2}$. Next, we choose $R_0$ so large that $|z + 2s\sqrt{t_0}|^2 \leq 8|t_0|s^2$ for all $s \geq R_0$ and

$$\frac{2C(\eta)}{\sqrt{\pi}} \int_{R}^{\infty} e^{-s^2/2}\, ds < \frac{\epsilon}{4}$$

for all $R \geq R_0$, where we have used the fact (cf. [20, p.201]) that $\int_R^{\infty} e^{-s^2}\, ds \leq \frac{e^{-R^2}}{2R}$ for $R > 0$. Let $\bar{B} = \{ w \mid |w| \leq r + 2R_0\sqrt{|t_0|} \}$. Since by assumption, $f_n \to f$ uniformly on compact subsets of $\mathbb{C}$, there is a positive integer $N$ such that $|f_n(w) - f(w)| < \epsilon/2$ for all $n \geq N$ and $w \in \bar{B}$. Hence, for $|z| \leq r$, using the preceding estimates along with the integral representation in Remark 3.5(3), we have for all $n \geq N$,

$$\begin{aligned}|e^{t_0D^2} f_n(z) - e^{t_0D^2} f(z)| &\leq \frac{1}{\sqrt{\pi}} \int_{-R_0}^{R_0} e^{-s^2} |f_n(z + 2s\sqrt{t_0}) - f(z + 2s\sqrt{t_0})|\, ds \\ &\quad + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-R_0} e^{-s^2} |f_n(z + 2s\sqrt{t_0}) - f(z + 2s\sqrt{t_0})|\, ds \\ &\quad + \frac{1}{\sqrt{\pi}} \int_{R_0}^{\infty} e^{-s^2} (|f_n(z + 2s\sqrt{t_0})| + |f(z + 2s\sqrt{t_0})|)\, ds \\ &\leq \frac{\epsilon}{2} + \frac{4}{\sqrt{\pi}} C(\eta) \int_{R_0}^{\infty} e^{-s^2/2}\, ds \\ &\leq \epsilon.\end{aligned}$$

□

The next two lemmas allow us to show that the choice of $m$ in Theorem 3.3 is independent of multiplication of $f$ by a (real) linear factor.

**Lemma 3.7.** *Let $f$ and $\phi$ be real entire functions. If $\phi(D)f(x)$ is entire, then for any $\alpha \in \mathbb{R}$, we have $\phi(D)(x + \alpha)f(x) = (x + \alpha)\phi(D)f(x) + \phi'(D)f(x)$.*

*Proof.* First consider the case $\phi(x) = x^n$ for some nonnegative integer $n$. Then $D^n(x + \alpha)f(x) = (x + \alpha)D^n f(x) + nD^{n-1} f(x)$. Using this and the Taylor expansion of $\phi$, the desired result follows. □

**Lemma 3.8.** *Let $h(x) \in \mathcal{L}\text{-}\mathcal{P}$. Then $h_1(x) = (x + \alpha)h(x) - \beta h'(x) \in \mathcal{L}\text{-}\mathcal{P}$ for all $\alpha \in \mathbb{R}$ and $\beta \geq 0$.*

*Proof.* Since it is known that the Laguerre-Pólya class is closed under differentiation (cf. [24]), we find that $-\beta D(e^{-(x+\alpha)^2/2\beta} h(x)) = e^{-(x+\alpha)^2/2\beta}[(x + \alpha)h(x) - \beta h'(x)]$, whence $h_1(x) \in \mathcal{L}\text{-}\mathcal{P}$. □

We shall denote the order of a function $f$ by $\rho(f)$ (see [3]).

**Lemma 3.9.** *Let $f(x) = p(x)\psi(x) \in \mathcal{L}\text{-}\mathcal{P}^*$, where $p(x) \in \mathbb{R}[x]$ and $\psi(x) \in \mathcal{L}\text{-}\mathcal{P}$, and the order $\rho(f) < 2$. Then there is a positive integer $m_0$ such that $e^{-mD^2} f(x) \in \mathcal{L}\text{-}\mathcal{P}$ for all $m \geq m_0$.*

*Proof.* By Theorem 2.4, there is a positive integer $m_0$ such that $e^{-mD^2} p(x) \in \mathcal{L}\text{-}\mathcal{P}$ for all $m \geq m_0$. If $\alpha \in \mathbb{R}$, then by Lemma 3.7, $e^{-mD^2}(x + \alpha)p(x) = (x + \alpha)e^{-mD^2} p(x) - 2mDe^{-mD^2} p(x) = (x+\alpha)h_m(x) - 2mh'_m(x)$, where $h_m(x) = e^{-mD^2} p(x)$. Hence by Lemma

3.8, we conclude that for all $m \geq m_0$, $\alpha \in \mathbb{R}$, the function $e^{-mD^2}(x+\alpha)p(x) \in \mathcal{L}\text{-}\mathcal{P}$. Now using Lemma 3.8 again and an easy induction argument shows that if $q(x)$ is a real polynomial having only real zeros, then

$$e^{-mD^2}q(x)p(x) \in \mathcal{L}\text{-}\mathcal{P}, \tag{3.5}$$

for all $m \geq m_0$.

Since $\psi(x) \in \mathcal{L}\text{-}\mathcal{P}$ of order less than 2, it follows from Lemma 3.4 that there is a sequence of real polynomials $\{\psi_n(x)\}_{n=1}^{\infty}$, where $\psi_n$ has only real zeros, such that $\psi_n \to \psi \quad (n \to \infty)$ uniformly on compact subsets of $\mathbb{C}$. Let $f_n(x) = p(x)\psi_n(x) \quad (n = 1, 2, \dots)$. Then $f_n \to f \quad (n \to \infty)$ uniformly on compact subsets of $\mathbb{C}$. By (3.5) above, for any positive integer $m$, $e^{-mD^2}f_n \to e^{-mD^2}f \quad (n \to \infty)$ uniformly on compact subsets of $\mathbb{C}$. Thus, using these facts and Hurwitz' theorem, we conclude that $e^{-mD^2}f(x) \in \mathcal{L}\text{-}\mathcal{P}$. □

**Theorem 3.10.** *Let $f \in \mathcal{L}\text{-}\mathcal{P}$ and suppose the order $\rho(f) < 2$. Let $u(x,t) = e^{-tD^2}f(x)$ for all $t > 0$. Then, for each fixed $t > 0$, we have $u(x,t) \in \mathcal{L}\text{-}\mathcal{P}$ and the zeros of $u(x,t)$ are all simple.*

*Proof.* Fix $t_0 > 0$. Then by Remark 3.5(3) , $u(x,t_0)$ is a real entire function of order less than 2. By the Hermite–Poulain theorem (see, for example, [25]) $u(x,t_0) \in \mathcal{L}\text{-}\mathcal{P}$. Thus, it suffices to show that the zeros of $u(x,t_0)$ are all simple. First suppose that for some $x_0 \in \mathbb{R}$,

$$u(x_0,t_0) = u_x(x_0,t_0) = 0, \tag{3.6}$$

but $u_{xx}(x_0,t_0) \neq 0$. Fix $0 < \epsilon < t_0$. By assumption, we have $u(x,t_0-\epsilon) \in \mathcal{L}\text{-}\mathcal{P}$. Now, it is known (cf. [33] or [10]) that any function $F(x) \in \mathcal{L}\text{-}\mathcal{P}$ satisfies the Laguerre inequalities:

$$L_k(F(x)) = (F^{(k)}(x))^2 - F^{(k-1)}(x)F^{(k+1)}(x) \geq 0, \tag{3.7}$$

for $k = 1, 2, 3, \dots$ and all $x \in \mathbb{R}$. To obtain the desired contradiction, consider

$$u(x,t_0-\epsilon) = u(x,t_0) + \epsilon u_{xx}(x,t_0) + O(\epsilon^2) \qquad (\epsilon \to 0)$$
$$u_x(x,t_0-\epsilon) = u_x(x,t_0) + \epsilon u_{xxx}(x,t_0) + O(\epsilon^2) \qquad (\epsilon \to 0)$$

and

$$u_{xx}(x,t_0-\epsilon) = u_{xx}(x,t_0) + \epsilon u_{xxxx}(x,t_0) + O(\epsilon^2) \qquad (\epsilon \to 0)\,.$$

Using equation (3.6), a calculation shows that $L_1(u(x_0,t_0)) = -\epsilon(u_{xx}(x_0,t_0))^2 + O(\epsilon^2)$. But then, for all $\epsilon$ sufficiently small, this expression is negative, whence $u(x,t_0) \notin \mathcal{L}\text{-}\mathcal{P}$, a contradiction.

Finally, suppose that for some $t_0 > 0$, $x_0$ is a zero of $u_(x,t_0)$ of multiplicity $k \geq 2$; that is, $\dfrac{\partial^j u(x_0,t_0)}{\partial x^j} = 0$, for $j = 0, 1, \dots, k$, but $\dfrac{\partial^{k+1} u(x_0,t_0)}{\partial x^{k+1}} \neq 0$. Let $v(x,t_0) = \dfrac{\partial^{k-1} u(x,t_0)}{\partial x^{k-1}}$. Then $v(x_0,t_0) = v_x(x_0,t_0) = 0$, but $v_{xx}(x_0,t_0) \neq 0$. Hence, as in the first part of the proof, we conclude that $v(x,t_0) \notin \mathcal{L}\text{-}\mathcal{P}$. Since the Laguerre-Pólya class is closed under differentiation, this is the desired contradiction, completing the proof of the theorem. □

**Corollary 3.11.** *Let $f \in \mathcal{L}\text{-}\mathcal{P}^*$ with $\rho(f) < 2$ and let $\alpha > 0$. Then $[e^{-\alpha D^2}]^m f(x) \in \mathcal{L}\text{-}\mathcal{P}$ with only simple zeros for all sufficiently large $m$.*

*Proof.* Combine Lemma 3.9 with Theorem 3.10. □

*Remark 3.12.* In the special case when $f(x)$ is the Riemann $\xi$-function, it was shown in [11, Corollary 1] that $e^{-\alpha D^2} f(x)$ has only simple real zeros for all $\alpha > 1/2$. (See also [5], where the operators $\cos \alpha D$ and $e^{-\alpha D^2}$, $\alpha > 0$, are applied to entire functions which can be represented by the Fourier transform of certain special kernels.)

**4. Simplicity of zeros.** In connection with his study of the distribution of zeros of the Riemann $\xi$-function, Pólya has shown (cf. [26, Hilfsatz III]) that if $q(x)$ is a polynomial possessing only real zeros and if $\phi(x)$ is a transcendental function in $\mathcal{L}\text{-}\mathcal{P}$, where $\phi(x)$ is not of the form $p(x)e^{\alpha x}$, where $p(x)$ is a polynomial, then the polynomial $\phi(D)q(x)$ has only simple real zeros. It seems natural to ask if this result can be extended to the situation where $q(x)$ is a transcendental entire function in the Laguerre-Pólya class. We are able to prove this whenever the canonical product in the representation of $\phi$ has genus zero or there is a bound on the multiplicities of the zeros of $q(x)$.

We begin with an analysis of the zeros of the functions $\phi(D)x^n = g_n^*(x)$ for $\phi \in \mathcal{L}\text{-}\mathcal{P}$. For convenience, we write the theorem in terms of $g_n(x) = x^n g_n^*(1/x)$. As noted above, Pólya has proved part (1) of the proposition. We provide a substantially different and unified proof of all the cases involved. This theorem also corrects and extends the results of [12].

**Proposition 4.1 (Simplicity of the zeros of Jensen polynomials).** *Let $\phi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k! \in \mathcal{L}\text{-}\mathcal{P}$ with $\phi(0) \neq 0$.*

(1) *If $\phi(x)$ is not of the form $p(x)e^{\beta x}$, where $p(x)$ is a polynomial and $\beta \neq 0$, then the zeros of the Jensen polynomials $g_n(x)$, $(n \geq 1)$, associated with $\phi(x)$ are all real and simple; consecutive zeros of $g_n$ of the same sign are separated by a zero of $g_{n-1}$. In particular, if $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}$ and all zeros have the same sign, then the zeros of $g_n$ and $g_{n-1}$ are interlacing.*

(2) *If $\phi(x) = p(x)e^{\beta x}$, where $p(x)$ is a polynomial and $\beta \neq 0$, then the zeros of $g_n(x)$ are simple for $n \leq \deg p$. For $n > \deg p$, $(\beta x + 1)^{n - \deg p}$ is a factor of $g_n(x)$.*

*Proof.* In order to cover two cases at once, let $m$ be the degree of $\phi$ if $\phi$ is a polynomial, and let $m$ be infinity otherwise. For any function $\phi \in \mathcal{L}\text{-}\mathcal{P}$, we have $\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} \geq 0$ for each $n \geq 1$. For $n \leq m < \infty$, $\phi(x) \neq ce^{\beta x}$ implies that, in fact,

$$\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} > 0 \tag{4.1}$$

by [**CC1**, Proposition 4.5].

(1) Since we shall be working with Sturm's theorem (see, for example, [24, p. 91], we adopt the usual notation of $V(a)$ for the number of sign changes in a sequence of polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ evaluated at $a$. We shall also write $V(+\infty)$ for the number of sign changes when $a$ is larger than all the zeros of all the polynomials in this sequence, and $V(-\infty)$ for the number of sign changes when $a$ is less than all the zeros.

We apply inequality (4.1) to

$$F(z) = e^z \phi(xz) = \sum_{n=0}^{\infty} g_n(x) \frac{z^n}{n!}, \tag{4.2}$$

where, for each fixed $x \in \mathbb{R}$, the function $F(z)$ is in $\mathcal{L}$-$\mathcal{P}$. This yields $\Delta_n(x) = g_n^2(x) - g_{n-1}(x)g_{n+1}(x) \geq 0$ for all $n$ and $x$. If $x \neq 0$, then $F(z)$ is not of the form $ce^{\beta z}$, so $\Delta_n(x) > 0$ for all $n$. If we now write $h_n(x) = \frac{1}{n!} x^n g_n(x^{-1})$, $x \neq 0$, and $\delta_n(x) = h_n^2(x) - h_{n-1}(x)h_{n+1}(x)$, a computation shows that $h_n'(x) = h_{n-1}(x)$ and $x^{2n}\Delta_n(x^{-1}) = (n-1)!(n+1)!\delta_n(x)$. Therefore $\delta_n(x) > 0$ for all $n \geq 1$ and all nonzero real $x$; from (4.1), the inequality $\delta_n(0) > 0$ also holds for $n \leq m$. From this we see that, for $n \leq m$, if $h_n(c) = 0$, then $h_{n-1}(c)$ and $h_{n+1}(c)$ have opposite signs, whence the polynomials $h_0(x), h_1(x), \ldots, h_n(x)$ form a Sturm sequence for $h_n(x)$. Since all $h_k(x)$, $k = 1, 2, \ldots$, have the same leading coefficient $a_0$, the numbers of sign changes are $V(-\infty) = k$ and $V(+\infty) = 0$, so that $h_k(x)$ has $k$ real zeros. The zeros are all simple since $h_k'(x) = h_{k-1}(x)$ and $h_k$ and $h_{k-1}$ have no common zeros as shown above.

If $\phi(z)$ is a polynomial, we must treat $x = 0$ more carefully to handle the case $n > m$. Each polynomial $h_n(x)$, for $n > m$ has a factor of $x^{n-m}$. We define the new polynomials $q_n(x) = h_n(x)$ for $n \leq m$ and $q_n(x) = h_n(x)/x^{n-m}$ for $n > m$. Then $q_n(0) = a_0/n! \neq 0$ and

$$\frac{n}{n+1} q_n^2(x) - q_{n-1}(x)q_{n+1}(x) = \frac{\delta_n(x)}{x^{2n-2m}} > 0$$

for $x \neq 0$, so that if $q_n(c) = 0$ then $q_{n-1}(c)$ and $q_{n+1}(c)$ have opposite signs. This sequence satisfies a generalized version of Sturm's theorem [24, §19], whence by [24, Satz 19.2], the number of real zeros of $q_n(x)$ is at least equal to the difference in the number of sign changes $V(-\infty) - V(+\infty) = n$ in the sequence $q_0(x), q_1(x), \ldots, q_n(x)$. Thus the zeros of $q_n(x)$ are all real. Again they are simple since they are the same as the nonzero roots of $h_n(x)$, which are simple as noted above. These zeros of $q_n(x)$ are the reciprocals of the zeros of $g_n(x)$, so $g_n(x)$ also has simple real zeros.

An easy computation shows that $ng_n(x) = ng_{n-1}(x) + xg_n'(x)$, from which we see that if $s < t$ are consecutive zeros of $g_n(x)$ of the same sign, then $n^2 g_{n-1}(s)g_{n-1}(t) = stg_n'(s)g_n'(t) < 0$, whence $g_{n-1}(x)$ has a zero in the interval $(s, t)$.

(2) If $\phi(x) = p(x)e^{\beta x}$, where $p(x)$ is a polynomial and $\beta \neq 0$, then, following part (1), $F(z)$ of (4.2) becomes a polynomial for $x = -\beta^{-1}$. Thus $\Delta_n(x) > 0$ only for $n \leq m = \deg p$, and simplicity is proved as above in this case. Now let $n > m$ and write $p(x) = \sum_{k=0}^{m} \gamma_k x^k$.

Then a computation shows that

$$g_n(x) = (\beta x + 1)^{n-m} \sum_{k=0}^{m} \gamma_k \frac{n!}{(n-k)!} x^k (\beta x + 1)^{m-k},$$

completing the proof of the theorem. □

Next we turn our attention to a generalization of Pólya's result [26, Hilfsatz III]. Theorem 3.10 handles the case where $\phi$ has a factor of $e^{-tx^2}$, $t > 0$, in its representation. Consequently, it suffices to consider functions $\phi$ of order less than two. We begin with a slight extension of the Hermite-Poulain theorem (cf. [25] or [24]).

**Lemma 4.2.** *Let $m$ be a positive integer. Set $\psi_m(x) = e^{\gamma x} \prod_{k=1}^{m}(x+\alpha_k)$, where $\gamma, \alpha_k \in \mathbb{R}$ for $k = 1, \ldots, m$. Let $f(x) \in \mathcal{L}\text{-}\mathcal{P}$ and, for each $m = 1, 2, \ldots$, let $h_m(x) = \psi_m(D)f(x)$. If $y_0$ is a zero of $h_m(x)$ of multiplicity at least two, then $y_0 + \gamma$ is a zero of $f(x)$ of multiplicity at least $m + 2$; that is, if $h_m(y_0) = h_m'(y_0) = 0$, then $f(y_0 + \gamma) = f'(y_0 + \gamma) = \cdots = f^{(m+1)}(y_0 + \gamma) = 0$.*

*Proof.* Since $e^{\gamma D} f(x) = f(x + \gamma)$, we may assume without loss of generality that $\gamma = 0$. Furthermore. we may assume that $f(x)$ is not of the form $ce^{\beta x}$, for some constant $c$, since in that case, $h_m(x) = c\psi_m(\beta)e^{\beta x}$, which has no zeros unless it is identically zero. Since $h_m(x) = \psi(D)f(x) = (e^{-\alpha_m x} D e^{\alpha_m x}) \cdots (e^{-\alpha_1 x} D e^{\alpha_1 x}) f(x)$ and the Laguerre-Pólya class is closed under differentiation, it follows that $h_m(x) \in \mathcal{L}\text{-}\mathcal{P}$. Recall from the proof of Theorem 3.10 that $f(x)$, being in $\mathcal{L}\text{-}\mathcal{P}$, must satisfy the Laguerre inequalities (3.7). Moreover, by examining $D\frac{f'}{f}(x)$, it is easily seen (cf. [10]) that if $f(x)$ is not of the form $ce^{\beta x}$, then $L_1(f(y_0)) = 0$ if and only if $y_0$ is a multiple zero of $f$; i.e. $f(y_0) = f'(y_0) = 0$.

We first consider the case when $m = 1$, so that $h_1(x) = \alpha_1 f(x) + f'(x)$. Then from $h_1(y_0) = h_1'(y_0) = 0$, we deduce that $L_1(f(y_0)) = 0$ and consequently that $f(y_0) = f'(y_0) = 0$. Since $h_1'(y_0) = 0$, we also have that $f''(y_0) = 0$, and thus the lemma holds for $m = 1$. For $m > 1$, we write $h_m(x) = (\alpha_m + D)h_{m-1}(x)$, and suppose that $h_m(y_0) = h_m'(y_0) = 0$. Then a repeated application of the argument above shows that $h_m(y_0) = h_m'(y_0) = h_{m-1}(y_0) = h_{m-1}'(y_0) = \cdots = f(y_0) = f'(y_0) = 0$. These, together with

$$\begin{aligned}
h_1(y_0) &= \alpha_1 f(y_0) + f'(y_0) = 0 \\
h_2(y_0) &= \alpha_2 h_1(y_0) + h_1'(y_0) \\
&= \alpha_1\alpha_2 f(y_0) + (\alpha_1 + \alpha_2) f'(y_0) + f''(y_0) = 0 \\
&\vdots \\
h_m(y_0) &= \alpha_m h_{m-1}(y_0) + h_{m-1}'(y_0) = 0,
\end{aligned}$$

imply the desired conclusion. □

*Remark 4.3.* In the usual formulation of the Hermite-Poulain theorem (cf. [19, p. 337], [24, p. 4] or [25]), it is assumed that the function $f$ in Lemma 4.2 is a polynomial. If $f$ is a polynomial in $\mathcal{L}$-$\mathcal{P}$, then the conclusion that any multiple zero of $h_m(D)f(x)$ is also a multiple zero of $f$ follows from standard counting arguments (cf. [24, p. 5]).

In order to prove our main result in this section when both $\phi$ and $f$ are transcendental functions in the Laguerre-Pólya class, we require the following lemmas (cf. [17, Lemma 1] in the special case when $\phi(x) = x$).

**Lemma 4.4.** *Let*

$$\phi(x) = cx^m e^{\beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

*where* $c, \beta, x_k \in \mathbb{R}$ *for* $k = 1, 2, \ldots,$ $m$ *is a nonnegative integer and* $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. *Suppose that* $\phi$ *has an infinite number of zeros. Let* $\phi_n(x) = \prod_{k=1}^{n}(1 + x/x_k)e^{-x/x_k}$ *for each* $n = 1, 2, \ldots$. *Then for any* $f \in \mathcal{L}$-$\mathcal{P}$ *and for any* $A > 0$, *there exists a positive integer* $N$ *such that* $\phi_n(D)f(x)$ *has only simple zeros in the interval* $I_n = (-A\sqrt{n}, A\sqrt{n})$ *for all* $n \geq N$.

*Proof.* Assume that the conclusion is false. Then, for some $A > 0$, there exists a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that for each $j$, $\phi_{n_j}(D)f(x)$ has a zero $y_j$ in the interval $I_{n_j}$, where the multiplicity of $y_j$ is at least two. But then, by Lemma 4.2, $t_j = y_j - \sum_{k=1}^{n_j} 1/x_k$ is a zero of $f(x)$ of multiplicity at least $n_j + 2$. Moreover, by passing to a subsequence of $\{t_j\}_{j=1}^{\infty}$, if necessary, we may assume that $t_i \neq t_j$ for $i \neq j$. We next claim that

$$\sum_{k=1}^{n} \frac{1}{|x_k|} = O(\sqrt{n}) \qquad (n \to \infty). \tag{4.3}$$

To see this, let $\tau$ denote the exponent of convergence (see, for example, [22, p. 285]) of the zeros $\{x_k\}_{k=1}^{\infty}$ of $\phi(x)$. Then it is known that $\tau = \overline{\lim}_{k\to\infty} \frac{\ln k}{\ln |x_k|}$ (cf. [22, Theorem 10.2]). Since $\phi \in \mathcal{L}$-$\mathcal{P}$, we have $\tau \leq 2$ and it follows that there is a positive integer $k_0$ such that $1/|x_k| \leq 1/\sqrt{k}$ for all $k \geq k_0$, from which an elementary integral estimate yields (4.3). Consequently, there is a constant $B > 0$ independent of $n$ such that $\sum_{k=1}^{n} 1/|x_k| \leq B\sqrt{n}$. Finally, to establish the desired contradiction, consider the nonzero roots $a_1, a_2, \ldots$ of $f(x)$, where $\sum_{j=1}^{\infty} 1/a_j^2 < \infty$. Then we have

$$\infty > \sum_{j=1}^{\infty} \frac{1}{a_j^2} > \sum_{j=1}^{\infty} \frac{n_j + 2}{t_j^2} > \sum_{j=1}^{\infty} \frac{n_j + 2}{(A + B)^2 n_j} = \infty,$$

our desired contradiction completing the proof. □

**Lemma 4.5.** *Let*

$$\phi(x) = \prod_{k=1}^{\infty}\left(1+\frac{x}{x_k}\right)e^{-\frac{x}{x_k}},$$

*where $x_k \in \mathbb{R}$ for $k = 1, 2, \ldots$ and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. For $n = 0, 1, \ldots,$ set*

$$R_n(x) = \prod_{j=n+1}^{\infty}(1+x/x_j)e^{-x/x_j} = \sum_{k=0}^{\infty} a_{k,n}x^k. \tag{4.4}$$

*If $f \in \mathcal{L}\text{-}\mathcal{P}$ and $f$ has order less than two, then as $n \to \infty$, the sequence $\{R_n(D)f\}$ converges uniformly to $f$ on compact subsets of $\mathbb{C}$.*

*Proof.* First we note that by Lemma 3.2, we have $R_n(D)f \in \mathcal{L}\text{-}\mathcal{P}$ for each $n = 0, 1, \ldots$. To estimate the Taylor coefficients in (4.4), we find, by Cauchy's inequality, that for each positive integer $n$,

$$|k!a_{k,n}| = |R_n^{(k)}(0)| \le \frac{k!}{s^k}M(s, R_n) \tag{4.5}$$

for any $s > 0$, $k = 0, 1, \ldots,$ where $M(s, R_n) = \max_{|z|=s}|R_n(z)|$. Since $\sum_{j=1}^{\infty} 1/x_j^2 < \infty$, there is a positive integer $n_0$ such that $\sum_{j=n+1}^{\infty} 1/x_j^2 < (6e)^{-1}$ for all $n \ge n_0$. Thus, using the familiar estimate for the logarithm of the modulus of the (Weierstrass) primary factors (see, for example, [19, p.11]) we have, for $n = 1, 2, \ldots$ and $|z| = s > 0$, the estimate $\log|R_n(z)| \le 6es^2\sum_{j=n+1}^{\infty} 1/x_j^2$. Hence, it follows that $M(s, R_n) \le e^{s^2}$ for all $s > 0$ and for all $n \ge n_0$. Setting $s = \sqrt{k}$, this inequality together with (4.5), yield the estimate

$$|a_{k,n}| \le \frac{e^k}{k^{k/2}} \text{ for all } n \ge n_0,\ k \ge 1. \tag{4.6}$$

Next, we fix $r > 0$ and $0 < \epsilon < (8e^2)^{-1}$. Then, as in the proof of Lemma 3.1 (with $\alpha_2 = 0$), a calculation shows that there is a positive integer $k_0$ such that for $|z| \le r$,

$$|f^{(k)}(z)| \le \frac{k!}{k^{k/2}}2^{k/2}e^k\epsilon^{k/2}e^{\epsilon r^2} \text{ for all } k \ge k_0.$$

Using this together with (4.6) and the elementary estimate $k! \le (k+2)k^ke^{-k}$, we obtain for $|z| \le r$ and for all $n \ge n_0$,

$$\varlimsup_{k\to\infty}\sqrt[k]{|a_{k,n}f^{(k)}(z)|} \le \varlimsup_{k\to\infty}\left[(k+2)^{1/k}\sqrt{2}e\sqrt{\epsilon}e^{\epsilon r^2/k}\right] = \sqrt{2}e\sqrt{\epsilon} \le \frac{1}{2}. \tag{4.7}$$

Using (4.7), we can find a positive integer $m_0$ such that for $|z| \le r$ and $n \ge n_0$,

$$\sum_{k=m_0+1}^{\infty}|a_{k,n}f^{(k)}(z)| < \frac{\epsilon}{3}. \tag{4.8}$$

Also, it follows from the uniform convergence (on compact subsets of $\mathbb{C}$) of the canonical product in (4.4), that $\lim_{n\to\infty} a_{0,n} = 1$ and $\lim_{n\to\infty} a_{k,n} = 0$ for $k \geq 1$. Therefore, there is a positive integer $n_1 \geq n_0$ such that for $|z| \leq r$, we have $|a_{0,n} - 1||f(z)| < \epsilon/3$ and $\sum_{k=1}^{m_0} |a_{k,n} f^{(k)}(z)| < \epsilon/3$ for all $n \geq n_1$. Finally, combining these two estimates with (4.8), we conclude that $|R_n(D)f(z) - f(z)| < \epsilon$ for $|z| \leq r$ and for all $n \geq n_1$. $\square$

*Remark.* An examination of the foregoing proof shows that Lemma 4.5 remains valid if $f \in \mathcal{L}\text{-}\mathcal{P}$ and $f$ has order two but its (normal) type is sufficiently small.

**Theorem 4.6.** *Let $\phi(x)$ and $f(x)$ be in $\mathcal{L}\text{-}\mathcal{P}$ with orders less than 2 and suppose that $\phi$ has an infinite number of zeros. If there is a bound on the multiplicities of the zeros of $f$, then $\phi(D)f(x)$ has only simple real zeros.*

*Proof.* The conditions on $\phi$ imply that it has a canonical representation of the form $\phi(x) = cx^m e^{\beta x} \prod_{k=1}^{\infty}(1 + x/x_k)e^{-x/x_k}$, where $c, \beta, x_k \in \mathbb{R}$ for $k = 1, 2, \ldots$, $m$ is a nonnegative integer and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. It is easy to see that we may assume that $c = 1$, $m = 0$ and $\beta = 0$. As we noted in the previous section, $\phi(D)f(x) \in \mathcal{L}\text{-}\mathcal{P}$ (see Lemma 3.2); thus to prove the theorem, it will suffice to show that for any fixed $A > 0$, the function $\phi(D)f(x)$ has only simple zeros in the interval $I_A = (-A, A)$.

Set $\theta_n(x) = \prod_{k=1}^{n}(1 + x/x_k)e^{-x/x_k}$ and $R_n = \prod_{k=n+1}^{\infty}(1 + x/x_k)e^{-x/x_k}$ for each $n = 1, 2, \ldots$. As in the proof of Lemma 4.4, we can find a positive number $B$ such that $\sum_{k=1}^{n} 1/|x_k| \leq B\sqrt{n}$ for all $n = 1, 2, \ldots$. By Lemma 4.4, there is a positive integer $N$ such that $\theta_n(D)f(x)$ has only simple zeros in the interval $J_n = (-(A + B)\sqrt{n}, (A + B)\sqrt{n})$ for all $n \geq N$. Fix $N_1 \geq N$ so large that

$$A + B \geq \left(\frac{A}{\sqrt{N_1}} + B\sqrt{\frac{p + N_1 - 2}{N_1}}\right), \tag{4.9}$$

where $p$ is the maximum multiplicity of any zero of $f$. The sequence $\{R_n(D)f(x)\}$ converges to $f(x)$ uniformly on compact subsets of $\mathbb{C}$ by Lemma 4.5. It then follows from Hurwitz' theorem that there is a positive integer $N_2 \geq N_1 + 1$ such that the entire function $R_{N_2}(D)[\theta_{N_1}(D)f(x)]$ has only simple zeros in the interval $J_{N_1}$. With the choices of $N_1$ and $N_2$ made above, set $F_{N_1,N_2}(x) = R_{N_2}(D)[\theta_{N_1}(D)f(x)]$, so that $\phi(D)f(x) = \psi(D)F_{N_1,N_2}(x)$, where $\psi(x) = \prod_{k=N_1+1}^{N_2}(1 + x/x_k)e^{-x/x_k}$.

Now suppose that $\phi(D)f(x)$ has a zero $y_0$ in $(-A, A)$, where the multiplicity of $y_0$ is at least two. Then by Lemma 4.2, $t_0 = y_0 - \sum_{k=N_1+1}^{N_2} 1/x_k$ is a zero of $F_{N_1,N_2}(x)$ of multiplicity at least $N_2 - N_1 + 2 \geq 3$. By our assumption on the multiplicities of the zeros of $f(x)$, we must have $N_2 \leq p + N_1 - 2$. From (4.9) and our choice of $B$, we obtain $|t_0| \leq |y_0| + \sum_{k=N_1+1}^{N_2} 1/|x_k| \leq A + B\sqrt{N_2} \leq A + B\sqrt{p + N_1 - 2} \leq (A + B)\sqrt{N_1}$. Hence $t_0$ is in the interval $J_{N_1}$. But this contradicts our choice of $N_1$ and $N_2$ so that $F_{N_1,N_2}(x)$ has only simple zeros in $J_{N_1}$. It follows that $\phi(D)f(x)$ can have only simple zeros in the (arbitrary) interval $(-A, A)$. $\square$

If the canonical product in the representation of $\phi$ has genus zero in Theorem 4.6, then the method of proof of this theorem, *mutatis mutandis*, also shows that we can dispense with the restriction on the multiplicities of the zeros of $f(x)$. The precise statement is as follows:

**Theorem 4.7.** *Let $\phi(x)$ and $f(x)$ be in $\mathcal{L}$-$\mathcal{P}$ with orders less than 2 and suppose that $\phi$ has an infinite number of zeros. If the canonical product in the representation of $\phi$ has genus zero, then $\phi(D)f(x)$ has only simple real zeros.*

*Open Problems 4.8.*

(1) We do not know whether or not the assumption (in Theorem 4.6) that there is a bound on the multiplicities of the zeros of $f$ is necessary. Thus, in connection with Theorems 4.6 and 4.7, the following problem remains open: If $\phi,\ f \in \mathcal{L}$-$\mathcal{P}$ and if $\phi$ has order less than two, then is it true that $\phi(D)f(x)$ has only *simple* real zeros?

(2) In [5], de Bruijn proved, in particular, that if $f$ is a real entire function of order less than two and if all the zeros of $f$ lie in the strip $S(\Delta) = \{\, z \in \mathbb{C} \mid |\Im z| \leq \Delta \,\}$ $\quad(\Delta \geq 0)$, then the zeros of $\cos(\lambda D)f(x)$ $\quad(\lambda \geq 0)$ satisfy $|\Im z| \leq \sqrt{\Delta^2 - \lambda^2}$, if $\Delta > \lambda$ and $\Im z = 0$, if $0 \leq \Delta \leq \lambda$. This result may be viewed as an analog of Jensen's theorem [21, §7] on the location of the nonreal zeros of the derivative of a polynomial. Is there also an analog of Jensen's theorem for $\phi(\lambda D)f(x)$, where $\phi$ is a more general function in the Laguerre-Pólya class?

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