APPROXIMATION THEORY:
A volume dedicated to Blagovest Sendov
2002, 1-24

Extremal Positive Trigonometric Polynomials

Dimitar K. Dimitrov *

In this paper we review various results about nonnegative trigonometric polynomials. The emphasis is on their applications in Fourier Series, Approximation Theory, Function Theory and Number Theory.

1. Introduction

There are various reasons for the interest in the problem of constructing nonnegative trigonometric polynomials. Among them are: Cesàro means and Gibbs' phenomenon of the the Fourier series, approximation theory, univalent functions and polynomials, positive Jacobi polynomial sums, orthogonal polynomials on the unit circle, zero-free regions for the Riemann zeta-function, just to mention a few.

In this paper we summarize some of the recent results on nonnegative trigonometric polynomials. Needless to say, we shall not be able to cover all the results and applications. Because of that this short review represents our personal taste. We restrict ourselves to the results and problems we find interesting and challenging.

One of the earliest examples of nonnegative trigonometric series is the Poisson kernel

$$1 + 2\sum_{k=1}^{\infty} \rho^k \cos k\theta = \frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}, \quad -1 < \rho < 1, \tag{1.1}$$

which, as it was pointed out by Askey and Gasper [8], was also found but not published by Gauss [30]. The problem of constructing nonnegative trigonometric polynomials was inspired by the development of the theory

*Research supported by the Brazilian science foundation FAPESP under Grant No. 97/06280-0 and CNPq under Grant No. 300645/95-3

of Fourier series and by the efforts for giving a simple proof of the Prime Number Theorem. This area of research reached its efflorescence in the beginning of the twentieth century due to the efforts of many celebrated mathematicians.

2. Fourier Series

Let ρ to tend to one in (1.1). We obtain the formal Fourier series

$$1 + 2\sum_{k=1}^{\infty} \cos k\theta \tag{2.2}$$

of the Dirac delta function $\delta(\theta)$. Now Poisson's formula

$$f(\rho e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi}) \frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2} d\varphi \tag{2.3}$$

and the theory of generalized functions strongly suggest that some sufficiently "good" functions can be recovered by its convolutions with certain kernels. It is already known what "good" and "certain" means but it was not so in the beginning of the nineteenth century, for instance, when Poisson himself, using his formula (2.3), produced a faulty convergence proof for Fourier series.

Let us concentrate on the properties of the kernels. It is intuitively clear from the above observation that these must inherit some of the properties of the Poisson kernel and of the Dirac delta functions. Then an *even positive kernel* is any sequence $\{k_n(\theta)\}$ of even, nonnegative continuous 2π-periodic functions if $k_n(\theta)$ are normalized by $(1/2\pi)\int_{-\pi}^{\pi} k_n(\theta)d\theta = 1$ and they converge uniformly to zero in any closed subset of $(0, 2\pi)$. Nowadays it is known that the convolutions

$$K_n(f,x) = k_n * f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(\theta) f(x-\theta)d\theta$$

of such kernels converge in $L_p[-\pi,\pi]$, $1 \le p \le \infty$, ($p = \infty$ means, of course, uniformly), almost everywhere and in mean provided the function $f(x)$ is 2π-periodic and belongs to an adequate space of functions (see [38]). More precisely, $f \in L_p[-\pi,\pi]$ yields the L_p convergence, the requirements that $f(x)$ is integrable in $[-\pi,\pi]$ guarantees the almost everywhere convergence, and when, in addition to the integrability of $f(x)$, for $x \in [-\pi,\pi]$, the limit $\lim_{h\to 0}(f(x+h)+f(x-h))$ exists, then

$$K_n(f;x) \longrightarrow (1/2)\lim_{h\to 0}(f(x+h)+f(x-h)) \quad \text{as } n \text{ diverges.}$$

To the best of our knowledge, Fejér [22] was the first to realize the above facts in 1900. He proved that the cosine polynomials

$$F_n(\theta) = 1 + 2\sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right)\cos k\theta \tag{2.4}$$

are nonnegative establishing the simple closed form

$$F_n(\theta) = \frac{\sin^2(n+1)\theta/2}{(n+1)\sin^2\theta/2}.$$

This immediately yields that $\{F_n(\theta)\}$ is a summability kernel. It is known that the corresponding convolutions

$$F_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-\theta)F_n(\theta)d\theta$$

coincide with the Cesàro means of the Fourier series of $f(x)$,

$$F_n(f;x) = \sigma_n(f;x) = \frac{S_0(f;x) + \cdots + S_n(f;x)}{n+1}.$$

Here $S_n(f;x)$ denotes the n-th partial sum of the Fourier series of $f(x)$.

Another reason for which Fejér was interested in nonnegative trigonometric sums is the Gibbs' phenomenon, called also the Gibbs-Wilbraham phenomenon. We refer the reader to Zygmund's classical book [63, Chapter II, §9] and a nice review paper of E. Hewitt and R. E. Hewitt [34] for more information about this topic. Fejér's interest led him to conjecture in 1910 that the partial sums $\sum_{k=1}^{n}(1/k)\sin k\theta$ of the sine Fourier series of $(\pi-\theta)/2,\ 0<\theta<\pi$, extended as an odd function, are positive in $(0,\pi)$.

Jackson [35] and Gronwall [33] proved Fejér's conjecture independently, but only within a few months. Nowadays the inequality

$$\sum_{k=1}^{n}\frac{\sin k\theta}{k} > 0 \quad \text{for} \quad \theta \in (0,\pi) \tag{2.5}$$

is called the Fejér-Jackson-Gronwall inequality.

In 1953 Turán [59] established the following important fact:

Theorem 1. *If*

$$\sum_{k=1}^{n} b_k \ \sin(2k-1)\theta \geq 0, \quad \theta \in (0,\pi),$$

then

$$\sum_{k=1}^{n} \frac{b_k}{k} \sin k\theta > 0 \quad \theta \in (0, \pi).$$

This theorem immediately yields the Fejér-Jackson-Gronwall inequality because it is very easy to prove that $\sum_{k=1}^{n} \sin(2k-1)\theta \geq 0$ for $\theta \in (0, \pi)$.

Recently Turán's result was used in [19] to provide some new nonnegative sine polynomials

Theorem 2. *The sine polynomials*

$$\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{k-1}{n(n+1)} \right) \sin k\theta, \tag{2.6}$$

$$\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{2(k-1)}{n^2-1} \right) \sin k\theta, \tag{2.7}$$

and

$$\sum_{k=1}^{n} \frac{1}{k} \sin k\theta \; - \; \frac{1}{2n} \sin n\theta,$$

are non-negative in $(0, \pi)$.

The graphs of the polynomials (2.6) and (2.7) show that their Gibb's phenomenon is much smaller than the Gibb's phenomenon of Fejér-Jackson-Gronwall's polynomials. Moreover, (2.6) and (2.7) approximate the function $(\pi - \theta)/2$ uniformly on every compact subset of $(0, \pi]$ much better than (2.5) does.

There are still many interesting nonnegative trigonometric polynomials which are related to Fourier series but we shall stop here in order to recall some well-known results concerning a general approach to obtaining such polynomials and review some of the interesting examples.

3. Construction of positive trigonometric polynomials

Fejér and Riesz (see Fejér's paper [24] of 1915) proved the following representation of nonnegative trigonometric polynomials: For every non-negative trigonometric polynomial $T(\theta)$,

$$T(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta),$$

there exists an algebraic polynomial $R(z) = \sum_{k=0}^{n} c_k z^k$ of degree n such that $T(\theta) = |R(e^{i\theta})|^2$. Conversely, for every algebraic polynomial $R(z)$ of degree n, the polynomial $|R(e^{i\theta})|^2$ is a nonnegative trigonometric polynomial of order n.

An equivalent form of the Fejér-Riesz representation is:

Theorem 3. *The trigonometric polynomial*

$$T(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta)$$

of order n is nonnegative for every real θ if and only if there exist complex numbers $c_k,\ k = 0, 1, \ldots, n$, such that

$$a_0 = \sum_{k=0}^{n} |c_k|^2, \quad a_k - ib_k = 2 \sum_{\nu=0}^{n-k} c_{k+\nu} \bar{c}_\nu, \quad \textit{for} \quad k = 1, \ldots, n.$$

Szegő [55] observed that in the Fejér-Riesz's representation of the nonnegative cosine polynomials the parameters c_k can be chosen to be real

Theorem 4. *Let*

$$C_n(\theta) = a_0/2 + \sum_{k=1}^{n} a_k \cos k\theta$$

be a cosine polynomial of order n which is nonnegative for every real θ. Then there exists an algebraic polynomial with real coefficients $R(z) = \sum_{k=0}^{n} c_k z^k$ of degree n such that $T(\theta) = |R(e^{i\theta})|^2$. Thus, the cosine polynomial $C_n(\theta)$ of order n is nonnegative if and only if there exist real numbers $c_k,\ k = 0, 1, \ldots, n$, such that

$$\begin{aligned} a_0 &= \sum_{k=0}^{n} c_k^2, \\ a_k &= \sum_{\nu=0}^{n-k} c_{k+\nu} c_\nu \quad \text{for} \quad k = 1, \ldots, n. \end{aligned}$$

The corresponding Fejér-Riesz type representation of the nonnegative sine polynomials was obtained recently in [19] and reads as follows:

Theorem 5. *The sine polynomial of order n*

$$S_n(\theta) = \sum_{k=1}^{n} b_k \sin k\theta$$

is nonnegative if and only if there exist real numbers $d_0, \dots, d_{n-1}$*, such that*

$$\begin{aligned} b_1 &= \sum_{k=0}^{n-1} d_k^2, \\ b_k &= \sum_{\nu=0}^{n-k} c_{k+\nu-1}c_\nu - \sum_{\nu=0}^{n-k-2} c_{k+\nu+1}c_\nu, \qquad \text{for} \quad k = 2, \dots, n-2, \\ b_{n-1} &= c_0c_{n-2} + c_1c_{n-1}, \\ b_n &= c_0c_{n-1}, \end{aligned}$$

where the parameters c *are represented in terms of the parameters* d *by*

$$c_k = ([k/2]+1) \sum_{\nu=0}^{[n/2]-[k/2]-(n,2)(k,2)} \sqrt{\frac{2}{([k/2]+\nu+1)([k/2]+\nu+2)}} d_{k+2\nu},$$

where $[s]$ *denotes the integer part of* s*, and* $(s,2)$ *is the rest of the division of* s *by* 2*.*

4. Some extremal positive trigonometric polynomials

Fejér-Riesz' type representations (see Theorems 3, 4 and 5 above) show that a variety of nonnegative trigonometric polynomials can be obtained by setting different values for the parameters c_k. However, the most natural and beautiful examples arise as solutions of some extremal problems for the coefficients of $T(\theta)$. For example, it is easily seen that Fejér's kernel (2.4) is the only solution of the extremal problem

$$\max\{a_1 + \cdots + a_n \ : \ 1 + 2\sum_{k=1}^{n} a_k \cos k\theta \geq 0\}.$$

Indeed, using the Fejér-Riesz representation for cosine polynomials, the above result is equivalent to the inequality between the arithmetic and the square means of the corresponding parametric sequence.

In 1915 Fejér [24] determined the maximum of $\sqrt{a_1^2 + b_1^2}$ provided $T(\theta)$ is nonnegative and $a_0 = 1$. He showed that

$$\sqrt{a_1^2 + b_1^2} \leq 2\cos(\pi/(n+2)) \tag{4.8}$$

and that this bound is sharp.

Szegő [54], in 1926, and independently Egerváry and Szász [21], in 1928, extended Fejér's result (4.8), finding estimates for the means $\sqrt{a_k^2+b_k^2}$, for $k=1,2,\ldots,n$, of nonnegative trigonometric polynomials, subject to $a_0=1$. More precisely, they proved that

Theorem 6. *If*

$$T(\theta)=1+\sum_{k=1}^{n}(a_k\cos k\theta+b_k\sin k\theta)$$

is nonnegative , then for any $k,\ 1\le k\le n$*,*

$$\sqrt{a_k^2+b_k^2}\le 2\cos\frac{\pi}{[n/k]+2}.$$

Moreover, equality is attained if and only if $T(\theta)$ *is of the form*

$$\tau(\theta)\left\{1+\frac{2}{p+2}\sum_{\nu=1}^{p}\left((p-\nu+1)\cos\nu\alpha+\frac{\sin(\nu+1)\alpha}{\sin\alpha}\right)\cos(\nu k(\theta-\psi))\right\},$$

where τ *is an arbitrary nonnegative trigonometric polynomial of order* q*,* $\alpha=\pi/(p+2)$*,* $p=[n/k]$*,* $n=pk+q$*,* $(0\le q<k)$ *and* ψ *is an arbitrary constant.*

In particular, one obtains, for any $k,\ 1\le k\le n$, the sharp estimates

$$|a_k|\le\cos\frac{\pi}{[n/k]+2}$$

for the coefficients of the nonnegative cosine polynomial

$$\frac{1}{2}+\sum_{k=1}^{n}a_k\cos k\theta.$$

It was pointed out in [19] that the nonnegative sine polynomial

$$\sum_{k=0}^{m}\sin(2k+1)\theta=\frac{\sin^2((m+1)\theta)}{\sin\theta},$$

which is a multiple of Fejér's kernel, turns out to be the only sine polynomial of odd order $2m+1$ with coefficient 1 of $\sin\theta$ for which the maximum of the moment $b_n,\ n=2m+1$, is attained.

There are only a few results about nonnegative sine polynomials with extremal coefficients. Since sine polynomials are odd functions, in what follows we shall call

$$S_n(\theta) = \sum_{k=1}^{n} b_k \ \sin k\theta$$

a nonnegative sine polynomial if $S_n(\theta) \geq 0$ for every $\theta \in [0, \pi]$.

In 1950 Rogosinski and Szegő [51] observed that, if $S_n(\theta)$ is non-negative, then $b_1 \geq 0$ and $b_1 = 0$ if and only if S_n is identically zero and considered the following extremal problem for nonnegative sine polynomials:

Problem 1. *Determine the minimum and maximum values of b_k provided the sine polynomial $S_n(\theta)$ is in*

$$\mathcal{S}_n = \left\{ S_n(\theta) = \sin\theta + \sum_{k=2}^{n} b_k \ \sin k\theta \quad : \quad S_n(\theta) \geq 0 \ \ for \ \ \theta \in [0, \pi] \right\}.$$

For each of these values, find the extremal sine polynomial which belongs to $\mathcal{S}_n$ and whose coefficient b_k coincides with the corresponding extremal value.

Rogosinski and Szegő suggested two methods for obtaining the extrema of the moments b_k and found the minimal and maximal values of b_2, b_3, b_{n-1} and b_n. More precisely, these limits are:

$$|b_2| \leq \begin{cases} 2\cos(2\pi/(n+3)), & n \text{ odd}, \\ 2\cos\theta_0, & n \text{ even}, \end{cases}$$

where θ_0 is the smallest zero of the function

$$(n+4)\sin((n+2)\theta/2) + (n+2)\sin((n+4)\theta/2);$$

$$|b_3 - 1| \leq 2\cos\frac{\pi}{[(n-1)/2]+3}, \qquad [(n-1)/2] \text{ even},$$

$$1 - 2\cos\theta_1 \leq b_3 \leq 1 + 2\cos\frac{\pi}{[(n-1)/2]+3}, \qquad [(n-1)/2] \text{ odd},$$

where θ_1 is the smallest zero of the function

$$([\frac{n-1}{2}]+4)\cos\frac{([(n-1)/2]+2)\theta}{2} + ([\frac{n-1}{2}]+2)\sin\frac{([(n-1)/2]+4)\theta}{2};$$

$$\begin{array}{ll} |b_{n-1}| \leq 1, & n \text{ odd}, \\ -(n-2)/(n+2) \leq b_{n-1} \leq 1, & n \text{ even}; \end{array}$$

and

$$\begin{array}{ll} -(n-1)/(n+3) \le b_n \le 1, & n \text{ odd}, \\ |b_n| \le n/(n+2), & n \text{ even}. \end{array}$$

However, in none of the cases the corresponding extremal sine polynomials were determined explicitly. The method suggested in [19] allowed us to obtain the extremal sine polynomials in various cases. Here are some of the results obtained in [19]:

Theorem 7. *Let $n = 2m + 2$ be an even positive integer. Then for every $S_n(\theta) \in \mathcal{S}_n$*

(i)

$$-\frac{m+1}{m+2} \le b_n \le \frac{m+1}{m+2}.$$

The equality $b_n = (m+1)/(m+2)$ is attained only for the nonnegative sine polynomial

$$\sum_{k=0}^{m} \left(\frac{(m+k+2)(m-k+1)}{(m+1)(m+2)} \sin(2k+1)\theta + \frac{(k+1)^2}{(m+1)(m+2)} \sin(2k+2)\theta \right).$$

The equality $b_n = -(m+1)/(m+2)$ is attained only for the nonnegative sine polynomial

$$\sum_{k=0}^{m} \left(\frac{(m+k+2)(m-k+1)}{(m+1)(m+2)} \sin(2k+1)\theta - \frac{(k+1)^2}{(m+1)(m+2)} \sin(2k+2)\theta \right).$$

(ii)

$$-\frac{m}{m+2} \le b_{n-1} \le 1.$$

The equality $b_{n-1} = 1$ is attained only for the nonnegative sine polynomials

$$\sin\theta + \sum_{k=1}^{m} (2pq \sin 2k\theta + \sin(2k+1)\theta) + pq \sin(2m+2)\theta,$$

where the parameters p and q satisfy $p^2 + q^2 = 1$.
The equality $b_{n-1} = -m/(m+2)$ is attained only for the nonnegative sine polynomials

$$\begin{aligned} \sin\theta \quad &+ \quad \sum_{k=1}^{m} 2pq \left(1 - \frac{2k^2}{m(m+2)}\right) \sin 2k\theta \\ &+ \quad \sum_{k=1}^{m} \left(1 - \frac{2k(k+1)}{m(m+2)}\right) \sin(2k+1)\theta \\ &- \quad pq \sin(2m+2)\theta, \end{aligned}$$

where p and q satisfy the relation $p^2 + q^2 = 1$.

Theorem 8. *Let* $n = 2m+1$ *be an odd positive integer. Then for every* $S_n(\theta) \in \mathcal{S}_n$

(i)

$$\frac{-m}{m+2} \leq b_n \leq 1.$$

The equality $b_n = 1$ *is attained only for the nonnegative sine polynomial*

$$\sum_{k=0}^{m} \sin(2k+1)\theta.$$

The equality $b_n = -m/(m+2)$ *is attained only for*

$$\sum_{k=0}^{m} \left((1 - \frac{2k(k+1)}{m(m+2)}) \sin(2k+1)\theta \right).$$

(ii)

$$-1 \leq \ b_{n-1} \leq 1.$$

The equality $b_{n-1} = 1$ *is attained only for the nonnegative sine polynomial*

$$\sum_{k=1}^{2m} \sin k\theta + \frac{1}{2} \sin(2m+1)\theta.$$

The equality $b_{n-1} = -1$ *is attained only for the nonnegative sine polynomial*

$$\sum_{k=1}^{2m} (-1)^{k+1} \sin k\theta + \frac{1}{2} \sin(2m+1)\theta.$$

In the same paper [51] of 1950 Rogosinski and Szegő found the highest possible derivative of the nonnegative sine polynomials at the origin. They proved that

$$s_n'(0) = 1+2b_2+\cdots+nb_n \leq \begin{cases} n(n+2)(n+4)/24, & n \text{ even}, \\ (n+1)(n+2)(n+3)/24, & n \text{ odd}, \end{cases} \tag{4.9}$$

provided b_k are the coefficients of a sine polynomial $s_n(\theta)$ in the space $\mathcal{S}_n$. The sine polynomials for which the above limits are attained were not determined explicitly. It was done recently in [2], where the main result reads as:

Theorem 9. *The inequality* (4.9) *holds for every $s_n(\theta) \in \mathcal{S}_n$. Moreover, if $n = 2m+2$ is even, then the equality $S'_{2m+2}(0) = (m+1)(m+2)(m+3)/3$ is attained only for the nonnegative sine polynomial*

$$S_{2m+2}(\theta) = \sum_{k=0}^{m} \left\{ \left(1 - \frac{k}{m+1}\right)\left(1 - \frac{k}{m+2}\right)\left(2k+1+\frac{k(k-1)}{m+3}\right)\sin(2k+1)\theta \right.$$
$$\left. +(k+1)\left(1 - \frac{k}{m+1}\right)\left(2 - \frac{k+3}{m+2} - \frac{k(k+1)}{(m+2)(m+3)}\right)\sin(2k+2)\theta \right\},$$

and, if $n = 2m+1$ is odd, the equality $S'_{2m+1}(0) = (m+1)(m+2)(2m+3)/6$ is attained only for the nonnegative sine polynomial

$$S_{2m+1}(\theta) = \sum_{k=0}^{m} \left\{ \left(1 - \frac{k}{m+1}\right)\left(1 + 2k - \frac{k(k+2)}{m+2} - \frac{k(k+1)(2k+1)}{(m+2)(2m+3)}\right)\sin(2k+1)\theta \right.$$
$$\left. +2(k+1)\left(1 - \frac{k}{m+1}\right)\left(1 - \frac{k+2}{m+2}\right)\left(1 + \frac{k+1}{2m+3}\right)\sin(2k+2)\theta \right\}.$$

The method developed in [19, 2] permitted the construction of various nonnegative cosine polynomials. Some of the interesting ones, obtained in [19], are:

$$1 + 2\sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right)\left(1 - \frac{2k(n+k+1)}{n(n+2)}\right)\cos k\theta, \tag{4.10}$$

$$1 + 2\sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right)\left(1 - \frac{k}{n+2}\right)\left(1 + \frac{k}{2n+3}\right)\cos k\theta, \tag{4.11}$$

$$1 + 2\sum_{k=1}^{n-1} \left(1 - \frac{k+1}{n+1/2}\right)\cos(k+1)\theta. \tag{4.12}$$

Sufficient conditions in terms of the coefficients in order that sine and cosine polynomial are positive were given by Vietoris [61] in 1958. He proved that

$$\sum_{k=1}^{n} a_k \cos k\theta > 0, \quad 0 < \theta < \pi,$$

and

$$\sum_{k=1}^{n} a_k \sin k\theta > 0, \quad 0 < \theta < \pi,$$

whenever $a_0 \ge a_1 \ge \ldots \ge a_n > 0$ and $2ka_k \le (2k-1)a_{2k-1}$. Askey and Steinig [9] provided a simpler proof of this result.

5. Approximation Theory

Since this volume is designed to experts in Approximation Theory, we outline only the principal application of the positive trigonometric polynomials to this area and pose an open problem.

The short discussion in Section 2 shows that the clue for constricting a very good explicit approximation tool for the 2π-periodic function is to find a "good" positive summability kernel. It was already mentioned in the beginning of the previous section that the most famous of the kernels, the Fejér's one, obeys certain extremal properties.

In 1912 Jackson [36, 37] proved his celebrated approximation result:

Theorem 10. *Let*

$$J_n(\theta) = \alpha_n F_n^2(\theta),$$

where α_n is chosen in such a way that $(1/2\pi)\int_{-\pi}^{\pi} J_n(\theta)d\theta = 1$. Then there exists a constant C, such that, for any $f \in C[-\pi, \pi]$ which is 2π-periodic,

$$\|f - J_n * f\|_\infty \leq C\omega(f; 1/n),$$

where $\| \ \|_\infty$ is the uniform norm in $[-\pi, \pi]$ and $\omega(f; \delta)$ denotes the modulus of continuity of $f(x)$ in the same interval.

Bernstein [10] (see also [58, Chapter 6]) proved that $\omega(f; 1/n)$ is the best possible rate of uniform approximation of continuous functions by trigonometric polynomials of order n.

There are different ways of proving Jackson's approximation. We are interested mainly in the constructive one. A nice simple proof of Jackson's theorem was given in Korovkin's [41] and Rivlin's [49, Chapter I] books. It uses essentially Fejér's result about the extremal nonnegative cosine polynomial of the form

$$1 + 2\sum_{k=1}^{n} a_k \cos k\theta,$$

whose coefficient a_1 is the largest possible, i.e. the cosine polynomial for which the equality in (4.8) is attained. It was mentioned in the previous section that Fejér's kernel is the nonnegative cosine polynomial of the above form whose sum of the coefficients is the largest possible. The nonnegative cosine sums (4.10), (4.11) and (4.12) also appeared as a consequence of the solution of some extremal problems concerning the coefficients of sine polynomials. It was proved in [19] that they are positive summability kernels. It would be of interest to find the best one.

Problem 2. *Denote by K the set of all positive summability kernels $\{k_n(\theta)\}_{n=0}^{\infty}$ and let*

$$\tilde{C} = \{f \in C[-\pi,\pi] \ : \ f(-\pi) = f(\pi), \ f(t) \neq const.\}.$$

Determine

$$\inf_{k_n \in K} \ \sup_{f \in \tilde{C}} \ \sup_{n \in I\!N} \frac{\|f - \ k_n * f\|_\infty}{\omega(f, 1/n)}$$

and the positive summability kernel for which the infimum is attained.

This problem might be called the problem about the best Jackson's constant for approximation by convolutions. It is worth mentioning that the smallest value of the constant in the Jackson inequality for approximation of periodic functions was found by Korneichuk [39] (see also [40]). Denote by $E_n(f)$ the best approximation of the function $f(\theta) \in \tilde{C}$ by trigonometric polynomials of order n, namely

$$E_n(f) = inf_{\tau \in \mathcal{T}_n} \|f - \tau\|,$$

where $\mathcal{T}_n$ is the set of the trigonometric polynomials of order n. Korneichuk proved in [39] that the smallest value of the of the constant M, independent of $f \in \tilde{C}$ and $n \in I\!N$, for which the inequalities

$$E_n(f) \leq M\omega(f; \pi/(n+1)), \quad f(\theta) \in \tilde{C} \ \ n = 1, 2, \ldots,$$

hold, is equal to 1. However, Problem 2 could be of interest on its own. The corresponding general question about the smallest value of the constant in Jackson's theorem for approximation by algebraic polynomials is one of the most challenging open problems in Approximation Theory. The approach suggested by Bojanov [12, 13] seems to trace a promising path towards solving this classical enigma.

6. Univalent functions and polynomials

In 1915 Alexander [1] proved that the polynomials

$$\sum_{k=1}^{n} a_k \frac{z^k}{k}$$

and

$$\sum_{k=1}^{n} a_{2k-1} \frac{z^{2k-1}}{2k-1}$$

are univalent in the unit disc of the complex plane provided $a_1 \geq \cdots \geq a_n > 0$.

In 1931 Dieudoneé [18] revealed the connection between Alexander's result and Fejér-Jackson-Gronwall's inequality. He proved that the polynomial

$$\sum_{k=1}^{n} a_k z^k$$

is univalent in the unit disc if and only if

$$\sum_{k=1}^{n} a_k z^{k-1} \frac{\sin k\theta}{\sin \theta} \neq 0 \quad \text{for} \quad |z| < 1,\ 0 \leq \theta \leq \pi.$$

In two consecutive papers, [25] of 1934, and [26] of 1936, Fejér extended the ideas of Alexander initiating the study of the so-called "vertically convex" functions. His approach provides one more connection between nonnegative trigonometric polynomials and univalent algebraic polynomials. This idea appeared in its final form in a joint paper of Fejér and Szegő [27] in 1951. The main result there states that an analytic function $f(z) = u(z) + iv(z)$ with real Maclaurin coefficients is univalent in the unit disc if $\partial u(\theta)/\partial\theta \leq 0$ for $0 \leq \theta \leq \pi$.

Another result which related positive trigonometric polynomials and univalent functions was obtained by Pólya and Schoenberg [47] in 1958. Before we formulate it, recall that the function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is called convex in the unit disc $D = \{z \ : \ |z| < 1\}$ if the image of D under $f(z)$ is a convex set. Pólya and Schoenberg proved that, if $f(z)$ is univalent and convex in D, then for any $n \in I\!N$, the polynomial

$$p_n(z) = \frac{1}{2\pi} \int_0^{2\pi} V_n(\theta - \xi) f(re^{i\xi}) d\xi, \quad z = re^{i\theta},$$

is also convex, where

$$V_n(\theta) = 1 + 2 \sum_{k=1}^{n} \frac{(n!)^2}{(n-k)!(n+k)!} \cos(k\theta) \geq 0$$

is the de la Vallée Poussin kernel.

We refer to the recent nice survey of Gluchoff and Hartmann [31], for these and other interesting results on the interplay of univalent functions and non-negative trigonometric series.

The above result of Fejér and Szegő can be reformulated immediately as follows:

Theorem 11. *If the sine polynomials with real coefficients*

$$\sin\theta + b_2 \sin 2\theta + \cdots + b_n \sin n\theta$$

is nonnegative in $[0, \pi]$*, then the polynomial*

$$z + \frac{b_2}{2} z^2 + \cdots + \frac{b_n}{n} z^n$$

is univalent in D*.*

Theorem 5 provides a complete characterization of the nonnegative sine polynomials in terms of the relations of the parameters b, a, c and d. Thus, it might be of interest to discuss some extremal problems for univalent polynomials. In particular, if we restrict ourselves to the subset of polynomials which satisfy the conditions of the latter theorem, the corresponding extremal problem would reduce to a question about extremal nonnegative sine polynomials. Some of the problems we find challenging follow.

Dieudoneé [18] found some upper bounds for the modulae of the coefficients of the polynomials in $\mathcal{U}_n$. Since $p_n'(z) \neq 0$ in D, then $|a_n| \leq 1/n$ and this upper bound is attained for the last coefficients of various univalent polynomials. Obviously none of the polynomials from $\mathcal{U}_n$ vanishes in D, except for the the simple zero at the origin, and the bound for a_n shows that the product of the modulae of the remaining zeros z_k, $k = 1, \ldots, n-1$, is at least n. Moreover, for the polynomials for which $|a_n| = 1/n$, at least one of the zeros must satisfy $|z_k| \leq n^{1/(n-1)}$. It would be curious to know which is the univalent polynomial of degree n with the smallest zero outside the unit disc. Formally we state

Problem 3. *For every* $n \in \mathbb{N}$*, determine*

$$\inf_{p_n \in \mathcal{U}_n} \inf\{|z_k| \ : \ p_n(z_k) = 0, \ z_k \neq 0\}$$

and the polynomial from $\mathcal{U}_n$ *for which the infimum is attained.*

Various extremal problems of this nature can be formulated. For example, one may look for the univalent polynomials, for which the sum the mudulae of the zeros, or the sum of the squares of the modulae of the zeros, is the smallest possible.

Another question is:

Problem 4. *Among all the polynomials from*

$$\mathcal{U}_n = \{p(z) = z + \alpha_2 z^2 + \cdots + \alpha_n z^n, \ \ p(z) \text{ is univalent in } D\}.$$

find the one with the smallest and with largest possible area, i.e., determine $\min\{A(p(D)) \ : \ p \in \mathcal{U}_n\}$ *and* $\max\{A(p(D)) \ : \ p \in \mathcal{U}_n\}$*, where* $A(p(D))$ *is the area of the image of* D *via* $p(z)$*.*

One of the reasons why this problem is natural is the celebrated Kobe (1/4)-theorem. It states that the image of the unit disc via any univalent function of the form $z + a_2 z^2 + \cdots$ contains the disc with radius $1/4$. Moreover, this result is sharp because for the Kobe function

$$k(z) := \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} k z^k$$

we have $k(-1) = -1/4$. Another interesting question which arises in connection with Kobe's (1/4)-theorem is:

Problem 5. *For any $n \in I\!N$ find the polynomial $p_n(z) \in \mathcal{U}_n$, for which the*

$$\inf\{|p_n(z)| \ : \ z = e^{i\theta}, \ 0 \le \theta < 2\pi\}$$

is attained.

Obviously the above infimums are bounded from below by $1/4$. Córdova and Ruscheweyh [14] studied the polynomial

$$q_n(z) = (1/\pi) \sum_{k=1}^{n} (n+1-k) \sin(k\pi/(n+1)) z^k.$$

It is univalent in D, $q_n(-1)$ goes to $-1/4$ and its coefficient a_1 goes to 1 as n diverges. Hence this polynomial solves the latter problem at least asymptotically. Is it the extremal one for every fixed n?

Relatively very recently, in 1994, Andrievskii and Ruscheweyh [3] proved the existence of a universal real constant $c > 0$ with the following property: For each f which is univalent in D and for any $n \ge 2c$ there exists a polynomial p_n of degree n, which is also univalent in D, with $f(0) = p_n(0)$ and

$$f(\kappa_n D) \subset p_n(D) \subset f(D), \quad \text{where} \ \ \kappa_n = 1 - c/n.$$

An important ingredient of the proof is a construction proposed by Dzjadyk [20] which involves powers of Jackson's kernel, or equivalently, even powers of the Fejér's kernel. A consideration of the Kobe function shows that $c \ge \pi$. Greiner [32] proved that $c < 73$ and stated that numerical experiments showed that the number 73 could eventually be replaced by 62. The problem of finding sharper bounds for c is of interest.

There is another, pretty unexpected connection between positive trigonometric sums and univalent functions. In 1912 W. H. Young [62] proved that the cosine polynomials

$$1 + \sum_{k=1}^{n} \frac{\cos k\theta}{k}$$

are nonnegative. Rogosinski and Szegő [50] considered inequalities of the form

$$\frac{1}{1+\alpha}+\sum_{k=1}^{n}\frac{\cos k\theta}{k+\alpha}\geq 0 \quad n=1,2,\ldots. \tag{6.13}$$

For $\alpha=0$ these reduce to Young's result and Rogosinski and Szegő showed the existence of a constant A, $1\leq A\leq 2(1+\sqrt{2})$, such that (6.13) hold for every α, $-1<\alpha\leq A$, and at least one of the latter inequalities fails for some x when $\alpha>A$. In 1969 G. Gasper [29] determined the exact value of A as the positive root of an algebraic equation of degree 7 and its numerical value is $A=4.56782$. Denote by $P_n^{(\alpha,\beta)}(x)$, $\alpha,\beta>-1$, the Jacobi polynomial of degree n, which is orthogonal in $(-1,1)$ to the polynomials of degree $n-1$ with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ and normalized by $P_n^{(\alpha,\beta)}(1)=(\alpha+1)_n/n!$. Here $(a)_k$ is the Pochhammer symbol. Bering in mind that

$$\frac{P_n^{(-1/2,-1/2)}(\cos\theta)}{P_n^{(-1/2,-1/2)}(1)}=\cos n\theta, \quad \frac{P_n^{(1/2,1/2)}(\cos\theta)}{P_n^{(1/2,1/2)}(1)}=\frac{\sin(n+1)\theta}{(n+1)\sin\theta},$$

and

$$\frac{P_n^{(1/2,-1/2)}(\cos\theta)}{P_n^{(1/2,-1/2)}(1)}=\frac{\sin(n+1/2)\theta}{(2n+1)\sin(\theta/2)},$$

one can conjecture that some of the known results on trigonometric polynomials may be generalized to sums of Jacobi polynomials. Fejér [23] was the first to do this. He proved that the Legendre polynomials $P_k(x)=P_k^{(0,0)}(x)$ satisfy the inequalities

$$\sum_{k=0}^{n}P_k(x)>0, \quad x\in[-1,1].$$

Feldheim [28] established the corresponding extensions for the Gegenbauer polynomials,

$$\sum_{k=0}^{n}\frac{P_k^{(\alpha,\alpha)}(x)}{P_k^{(\alpha,\alpha)}(1)}>0, \quad x\in[-1,1], \quad \alpha\geq 0.$$

In the sixties and the seventies of the previous century Askey and his coauthors extended most of the above mentioned classical trigonometric polynomials obtaining various results on positive sums of Jacobi polynomials. It turned out that one of the natural candidates to be such a positive sum, for various values of the parameters α and β, is

$$D_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)}, \quad x\in[-1,1].$$

The reason why the parameters α and β interchange in the numerator and denominator comes from the problem of constructing positive quadrature rules and a detailed explanation was given by Askey in [6]. In 1976 Askey and Gasper [7] proved, among the other results, that $D_n^{(\alpha,\beta)}(x) \geq 0,\ x \in [-1,1]$ provided $\beta = 0$ and $\alpha \geq -2$. In 1984 Louis de Branges [15] used this result, in the particular case when α is an even integer, in the final stage of his celebrated proof of the Bieberbach conjecture. Recall that the latter result, which might be called de Branges-Bieberbach theorem, states that the Maclaurin coefficients of any function

$$z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots,$$

which is univalent in the unit disc, satisfy $|a_n| \leq n$ for every $n \in \mathbb{N}$. We refer to the paper of Askey and Gasper [8] for more information about positive sums of Jacobi polynomials.

7. Number Theory

Positive trigonometric polynomials played an essential role in some important problems both in analytic and algebraic number theory.

Probably the most famous use of such polynomials was made by de la Vallée Poussin [16] in his proof of the Prime number theorem. Recall that, if $\pi(x)$ denotes the number of primes less than x, the Prime number theorem states that

$$\frac{\pi(x)\log x}{x} \to 1 \quad \text{as} \quad n \text{ diverges.}$$

In order to prove it, de la Vallée Poussin needed to verify that the Riemann zeta-function $\zeta(z)$ does not vanish on the boundary of its critical strip $0 < Re(z) < 1$. For this purpose he used the simple inequality

$$3 + 4\cos\theta + \cos 2\theta \geq 0, \quad \theta \in \mathbb{R}.$$

Later on de la Vallée Poussin [17] made another use of this inequality to prove that

$$\zeta(\sigma + it) \neq 0 \ \text{ for } \ \sigma \geq 1 - (R\log t)^{-1} \text{ and } t \geq T \tag{7.14}$$

with $T = 12$ and certain large value of R and

$$\pi(x) - \int_2^\infty (\log u)^{-1} du = \mathcal{O}(x\exp(-K\sqrt{\log x})) \tag{7.15}$$

with $K = 0.186$. Landau [43, 44] developed further the ideas of de la Vallée Poussin aiming to describe subregions of the critical strip which are free of zeros of $\zeta(z)$ and to obtain better limits for the error in the prime number theorem. Denote

$$\mathcal{C}_n^+ = \{c_n(\theta) = \sum_{k=0}^{n} a_k \cos k\theta \geq 0, \quad a_k \geq 0,\ k = 0, \dots, n, \quad a_0 < a_1\}.$$

Landau considered the quantities

$$V(c_n) = \frac{c_n(0) - a_0}{(\sqrt{a_1} - \sqrt{a_0})^2} = \frac{a_1 + \cdots + a_n}{(\sqrt{a_1} - \sqrt{a_0})^2}$$

and formulated the following interesting extremal problem about nonnegative cosine sums:

Problem 6. *For any $n \in I\!N$ determine*

$$V_n = \inf\{V(c_n) \ : \ c_n(\theta) \in \mathcal{C}_n^+\}.$$

Find the limit $V_\infty = \lim_{n\to\infty} V_n$.

Landau justified the importance of this problem establishing two fundamental results. The first one concerns a zero-free region for the Riemann ζ-function. It states that the inequality (7.14) holds for any $\varepsilon > 0$ with $R = V_\infty/2 + \varepsilon$ and $T = T(\varepsilon)$. The second result of Landau states that (7.15) remains true for any $K < \sqrt{2/V_\infty}$.

An auxiliary question related to the latter problem is:

Problem 7. *Let*

$$V(c_n) = \frac{c_n(0)}{a_1 - a_0} = \frac{a_0 + \cdots + a_n}{a_1 - a_0}.$$

For any $n \in I\!N$, determine

$$U_n = \inf\{U(c_n) \ : \ c_n(\theta) \in \mathcal{C}_n^+\}.$$

Find the limit $U_\infty = \lim_{n\to\infty} U_n$.

The behaviour of the sequence has attracted the interest of many celebrated mathematicians. Landau [45] calculated U_2 and U_3, and Tchakaloff [56, 57] found the values of U_n for $n = 4, 5, 6, 7, 8, 9$. Landau and Schur (see [44]) and Van der Waerden [60] obtained lower bounds for U_∞ which are close to U_9, calculated by Tchakaloff. Stechkin [53] established the limits $32.49 < V_\infty < 34.91$. Since then these limits have been improved many times by

Reztsov [48], Arestov [4], Arestov and Kondratev [5]. To the best of our knowledge the sharpest known limits are $34.4689 < V_\infty < 34.5036$ and are due to Arestov and Kondratev [5].

Though Stechkin wrote in [53] that the possibilities for extending the zero-free regions for $\zeta(s)$ had been exhausted, we think that the problem of determining the exact values of V_∞ is still challenging in itself.

The complex number α is called an algebraic integer of degree n if it is a zero of a monic algebraic polynomial of degree n with integer coefficients, say

$$p_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0, \quad a_k \text{ are integers.}$$

Suppose that $p_n(z)$ is the polynomial of the smallest degree with this property and denote by $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ the conjugates of α, i.e., all the zeros of $p_n(z)$. Introduce the denotations

$$|\alpha|_\infty = \max_{1\le j\le n} |\alpha_j|, \quad |\alpha|_1 = \prod_{j=1}^{n} \max\{1, |\alpha_j|\}.$$

The quantity $|\alpha|_1$ is sometimes called the Mahler measure of $p_n(z)$ and is denoted by $M(f)$. In 1857 Kronecker [42] proved that $|\alpha|_\infty = 1$ if and only if α is a root of unity. There are another two problems which had been motivated by Kronecker's result. In 1933 Lehmer [46] asked if there exists a constant $c_1 > 0$ such that the condition $|\alpha|_1 \le 1 + c_1$ implies that α is a root of unity, and in 1965 Schinzel and Zessenhaus [52] posed the question of the existence of a constant $c_\infty > 0$ such that $|\alpha|_\infty \le 1 + c_\infty/n$ yields that α is a root of unity. It is not difficult to see that the statement of Lehmer's conjecture implies the statement of the Schinzel-Zessenhaus conjecture. In 1971 Blanksby and Montgomery [11] made an important contribution towards the answers of these questions. They proved that each of the conditions

$$|\alpha|_\infty \le 1 + (30n^2 \log 6n)^{-1} \quad \text{and} \quad |\alpha|_1 \le 1 + (52n \log 6n)^{-1}$$

implies that α is a root of unity. Blanksby and Montgomery provided also a new proof of Dirichlet's theorem and used the positivity of the Fejér kernel in their proofs. It has been of interest to determine the extremal non-negative cosine polynomial which would allow improvements of the above results.

References

[1] J. W. Alexander, Functions which map the interior of the unit disc upon simple regions, *Ann. of Math.* **17** (1915-16), 12–22.

[2] R. Andreani and D. K. Dimitrov, An extremal nonnegative sine polynomial, *Rocky Mountain J. Math.* (to appear)

[3] V. V. Andrievskii and St. Ruscheweyh, Maximal polynomial subordination to univalent functions in the unit disc, *Constr. Approx.* **10** (1994), 131-144.

[4] V. V. Arestov, On extremal properties of the nonnegative trigonometric polynomials, *Trans. Inst. Mat. Mekh. Ekaterinburg* **1** (1992) 50-70 [In Russian].

[5] V. V. Arestov and V. P. Kondratev, On an extremal problem for nonnegative trigonometric polynomials, *Mat. Zametki* **47** (1990), 15-28 [In Russian].

[6] R. Askey and G. Gasper, Positive quadrature methods and positive polynomial sums, *in* "Approximation Theory, V", pp. 1–29, Academic Press, Boston, 1986.

[7] R. Askey and G. Gasper, Positive Jacobi polynomial sums, *Amer. J. Math.* **98** (1976), 709–737.

[8] R. Askey and G. Gasper, Inequalities for polynomials, *in* "The Bieberbach Conjecture: Proceedings of the Symposium on the Occasion of the Proof", (A. Baernstein II et al., Eds.), pp. 7–32, Amer. Math. Soc., Providence, RI, 1986.

[9] R. Askey and J. Steinig, Some positive trigonometric sums, *Trans. Amer. Math. Soc.* **187** (1974), 295–307.

[10] S. N. Bernstein On the best approximation of continuous functions by polynomials of given degree, *in* "Collected Papers", Vol. I, pp. 11-104, 1912.

[11] P. E. Blanksby and H. L. Montgomery, Algebraic integers near the unit circle, *Acta Arithmetica* **18** (1971), 355–369.

[12] B. D. Bojanov, A Jackson type theorem for Tchebycheff systems, *Math. Balkanica (N.S.)* **10** (1996), 73–82.

[13] B. D. Bojanov, Remarks on the Jackson and Whitney constants, *in* "Recent progress in inequalities (Niš, 1996)", pp. 161–174, Math. Appl., **430**, Kluwer Acad. Publ., Dordrecht, 1998.

[14] A. Córdova and St. Ruscheweyh, On the maximal range problem for slit domains, *in* "Computational Methods and Function Theory, 1989" (St. Ruscheweyh, E. B. Saff, S. L. Salinas, and R. S. Varga, Eds.), pp. 33-44, Lecutre Notes in Mathematics **1435**, Springer Verlag, New York, Berlin, 1983.

[15] L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.* **154** (1985), 137–152.

[16] de la Vallée Poussin, Recherches analitiques sur la théorie des nombres premiers, *Ann. Soc. Sci. Bruxelles* **20** (1896), 183–256, 281–297.

[17] de la Vallée Poussin, Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieurs à une limite donnée, *Mem. Acad. Royal Sci. Lett. Beaux-Arts Belg.* **59** (1899), 1–74.

[18] J. Dieudoneé, Recherches sur quelques problèmes relatifs aux polynômes et aux fonctions bornées d'une variable complexe, *Ann. Sci. École Norm. Sup.* (3) **48** (1931), 247–358.

[19] D. K. Dimitrov and C. A. Merlo, Nonnegative trigonometric polynomials *Constr. Approx.* **18** (2001), 117–143.

[20] V. K. Dzjadyk, On the approximation of continuous functions in closed domains with corners, and on a problem of S. M. Nikolskii, I, *Izv. Akad Nauk SSSR Ser. Math.* **26** (1962), 797–824 [In Russian]; English translation: *Amer. Math. Soc. Transl.* **53** (1966), 221–252.

[21] E. Egerváry and Szász, Einige Extremalprobleme im Bereiche der trigonome-tri-schen Polynome, *Math. Z.* **27** (1928), 641–652.

[22] L. Fejér, Sur les functions bornées et integrables, *C. R. Acad. Sci. Paris* **131** (1900), 984–987.

[23] L. Fejér, Sur le développement d'une fonction arbitraire suivant les fonctions de Laplace, *C. R. Acad. Sci. Paris* **146** (1908), 224–227.

[24] L. Fejér, Über trigonometriche Polynome, *J. Reine Angew. Math.* **146** (1915), 53–82.

[25] L. Fejér, On new properties of the arithmetical means of the partial sums of Fourier series, *J. Math. Physics* **13** (1934), 83–88.

[26] L. Fejér, Untersuchungen über Potenzreihen mit mehrfach monotoren Koeffizientenfolge, *Litt. Acad. Sci. Szeged* **8** (1936), 89–115.

[27] L. Fejér and G. Szegő, Special conformal maps, *Duke Math. J.* **18** (1951), 535–548.

[28] E. Feldheim, On the positivity of certain sums of ultraspherical polynomials, *J. Anal. Math* **11** (1963), 275–284.

[29] G. Gasper, Nonnegative sums of cosine, ultraspherical and Jacobi polynomials, *J. Math. Anal. Appl.* **26** (1969), 60–68.

[30] K. F. Gauss, Développement des fonctions príodiques en séries, *in* Werke VII, pp. 469–172, K. Gesell. Wiss., Göttingen, 1906.

[31] A. Gluchoff and F. Hartmann, Univalent polynomials and non-negative trigonometric sums, *Amer. Math. Monthly* **105** (1998), 508–522.

[32] R. Greiner, "Zur Güte der Approximation shlichter Abbildungen durch maximal subordinierende Polynomfolgen", Diploma thesis, Würzburg, 1993.

[33] T. H. Gronwall, Über die Gibbssche Erscheinung und die trigonometrischen Summen $\sin x + \frac{1}{2}\sin 2x + \cdots + \frac{1}{n}\sin nx$, *Math. Ann.* **72** (1912), 228–243.

[34] E. Hewitt and R. E. Hewitt, The Gibbs-Wilbraham phenomenon: an episode in Fourier analysis, *Arch. Hist. Exact Sci.* **21** (1979), 129–160.

[35] D. Jackson, Über eine trigonometrische Summe, *Rend. Circ. Mat. Palermo* **32** (1911), 257–262.

[36] D. Jackson, Über die Genauigkeit der Annäherung stetiger Funktionen durch rationale Funktionen gegebenen Grades und trigonometrische Summen gegebenen Ordnung, doctoral thesis, University of Göttingen, 1912.

[37] D. Jackson, "The theory of approximations", Amer. Math. Soc. Colloq. Publ., **vol. XI**, Providence, R. I.: Amer. Math. Soc., 1930.

[38] Y. Katznelson, "An Introduction to Harmonic Analysis", John Wiley, New York, 1968.

[39] N. P. Korneichuk, The exact constant in the theorem of D. Jackson on the best uniform approximation to continuous periodic functions, *Dokl. Akad. Nauk SSSR* **145** 1962, 514–515 [In Russian].

[40] N. P. Korneichuk, The exact constant in the Jackson inequality for continuous periodic functions, *Mat. Zametki* **32** (1982), 669–674 [In Russian].

[41] P. P. Korovkin, "Operators and Approximation Theory", Fizmatgiz, Moscow, 1959 [In Russian] English translation: Hindustan Publishing Co., Delhi, 1960.

[42] L. Kronecker, Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten, *J. Reine Angew. Math.* **53** (1857), 173–175.

[43] E. Landau, Beitrage zur analytischen Zahlentheorie, *Rend. Circ. Mat. Palermo* **26** (1908), 169–302.

[44] E. Landau, "Handbuch der Lehre von der Verteilung der Primzahlen", Druk and Verlag von B. G. Teubner, Leipzig, Berlin, 1909.

[45] E. Landau, Nachtrang zu meiner Arbeit "Eine Frage über trigonometrishce Polynome", *Ann. Scuola Norm. Sup. Pisa*, II, **5** (1936), 141.

[46] D. H. Lehmer, Factorization of certain cyclotomic functions, *Ann. Math.* (2) **34** (1933), 461–479.

[47] G. Pólya and I. J. Schoenberg Remarks on de la Vallée Poussin maens and convex conformal maps of the circle, *Pacific J. Math.* **8** (1958), 295–334.

[48] A. V. Reztsov, Some extremal properties of nonnegative triginometric polynomials, *Mat. Zametki* **39** (1986), 245–252 [In Russian]; English translation: *Math. Notes* **39** (1986), 133-137.

[49] T. J. Rivlin, "An Introduction to the Approximation of Functions", Dover Publ., New York, 1981.

[50] W. W. Rogosinski and G. Szegő, Über die Abschnitte von Potenzreihen, die in einem Kreise beschränkt bleiben, *Math. Z.* **28** (1928), 73–94.

[51] W. W. Rogosinski and G. Szegő, Extremum problems for non-negative sine polynomials, *Acta Sci. Math. Szeged* **12** (1950), 112–124.

[52] A. Schinzel and H. Zessenhaus, A refinement of two theorems of Kronecker, *Mich. Math. J.* **12** (1965), 81–84.

[53] S. B. Stechkin, Some extremal properties of positive trigonometric polynomials, *Mat. Zametki* **7** (1970), 411–422; English translation: *Math. Notes* **7** (1970), 248–255.

[54] G. Szegő, Koeffizientenabschätzungen bei ebenen und räumlichen harmonischen Entwicklungen, *Math. Ann.* **96** (1926/27), 601–632.

[55] G. Szegő, "Orthogonal Polynomials", 4th ed., Amer. Math. Soc. Coll. Publ. **23**, Providence, RI, 1975.

[56] L. Tchakaloff, Minimal properties of some trigonometric polynomials, *Godishnik Sofiisk Univ. Fiz Mat. Fak,* **19** (1922-23), 365–388.

[57] L. Tchakaloff, Trigonometrische Polynome mit einer Minimumeigenschaft. *Ann. Scuola Norm. Sup. Pisa*, II, **9** (1940), 13–26.

[58] A. F. Timan, "Theory of Approximation of Functions of a Real Variable", Fizmatfiz, Moscow, 1960 [In Russian]; English translations: Pergammon Press, Oxford, 1963; and: Dover Publ., New York, 1994.

[59] P. Turán, On a trigonometric sum, *Ann. Soc. Polon. Math.* **25** (1953), 155–161.

[60] B. L. Van der Waerden, Über Landau's Rewies de Primzahlsatzen, *Math. Z.* **52** (1949), 649–653.

[61] L. Vietoris, Über das Vorzeichechen gewisser trigonometrischer Summen, *Sitzungsber. Öst. Akad. Wiss.* **167** (1958), 125–135; *Anzeiger. Öst. Akad. Wiss.* (1959) 192–193.

[62] W. H. Young, On certain series of Fourier, *Proc. London Mah. Soc.* (2) **11** (1912), 357–366.

[63] A. Zygmund, "Trigonometric series", Cambridge Univ. Press, Cambridge, 1968.

Dimitar K. Dimitrov

Departamento de Ciências de Computação e Estatística
IBILCE, Universidade Estadual Paulista,
15054-000 São José do Rio Preto, SP
BRAZIL
e-mail: dimitrov@dcce.ibilce.unesp.br