NONNEGATIVE TRIGONOMETRIC POLYNOMIALS

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ABSTRACT. An extremal problem for the coefficients of sine polynomials, which are nonnegativein $[0, \pi]$, posed and discussed by Rogosinski and Szegő is under consideration. An analog of the Fejér-Riesz representation of nonnegative-general trigonometric and cosine polynomials is proved for nonnegativesine polynomials. Various extremal sine polynomials for the problem of Rogosinski and Szegő are obtained explicitly. Associated cosine polynomials $k_n(\theta)$ are constructed in such a way, that $\{k_n(\theta)\}$ are summability kernels. Thus, the L_p, pointwise and almost everywhere convergence of the corresponding convolutions is established.

1. INTRODUCTION

Among the various reasons for the interest in the problem of constructing non-negativetrigonometric polynomials are: the Gibbs phenomenon [19, Chapter II, §9], univalent functions and polynomials [8], positive Jacobi polynomial sums [3], orthogonal polynomials on the unit circle [18], zero-free regions for the Riemann zeta-function [1, 2], just to mention a few.

Our interest in this subject comes from the classical Approximation Theory. In this paper we construct some new positive summability kernels. Recall that the sequence $\{k_n(\theta)\}$ of even, nonnegativecontinuous 2π-periodic functions is called an *even positive kernel* if $k_n(\theta)$ are normalized by $(1/2\pi)\int_{-\pi}^{\pi} k_n(\theta)d\theta = 1$ and they converge to zero in any closed subset of $(0, 2\pi)$. It is classically known that the convolutions of such kernels with 2π-periodic functions $f \in L_p[-\pi, \pi]$ converge to f in the L_p-norm, for $1 \le p \le \infty$. To the best of our knowledge, Fejér [6] was the first to construct such a kernel. He proved that

$$F_n(\theta) = 1 + 2\sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right)\cos k\theta \tag{1.1}$$

are nonnegative, and established the uniform convergence of the corresponding convolutions with continuous functions. These convolutions are nothing but the Cesàro means of the Fourier series. Jackson [10, 11] used the kernel $J_n(\theta) = F_n^2(\theta)$ to prove his celebrated approximation theorem.

The basic tool for constructing nonnegativetrigonometric polynomials $T(\theta)$ is Fejér and Riesz' (see [7]) theorem which states that $T(\theta)$ is nonnegativeif and only if there exists an algebraic polynomial $R(z)$, such that $T(\theta) = |R(e^{i\theta})|^2$. However,

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most of the positive summability kernels turn out to be solutions of some extremal problems for the coefficients of $T(\theta)$. Fejér's kernel (1.1) itself is the only nonnegativecosine polynomial of the form $1+2\sum_{k=1}^{n} a_k \cos k\theta \geq 0$ with the maximal possible sum of its coefficients. A nice simple proof of Jackson's theorem in Rivlin [12, Chapter I] is based on the nonnegativecosine polynomial whose first coefficient a_1 is the largest possible, provided $a_0 = 1$. This polynomial was determined explicitly by Fejér [7].

Szegő [14] and Egerváry and Szász [5] extended this result of Fejér, determining the maximums of the means $\sqrt{a_k^2+b_k^2}$, for $k = 1, 2, \ldots, n$, of nonnegativetrigonometric polynomials whose coefficient a_0 is fixed to be 1. Their estimates yield lower and upper limits for the coefficients a_k of the nonnegativecosine polynomial

$$1+\sum_{k=1}^{n} a_k \cos k\theta.$$

In all these cases the corresponding extremal polynomials were found explicitly. However, very little is known about the extremal values of the coefficients of the nonnegativesine polynomials.

Since sine polynomials are odd functions, in what follows we shall call

$$S_n(\theta) = \sum_{k=1}^{n} b_k \ \sin k\theta$$

a nonnegativesine polynomial if $S_n(\theta) \geq 0$ for every $\theta \in [0, \pi]$. It is clear that, if $S_n(\theta)$ is nonnegative, then $b_1 \geq 0$ and $b_1 = 0$ if and only if S_n is identically zero. Motivated by the results for the general trigonometric and for the cosine polynomials described above and bearing in mind this observation about b_1, Rogosinski and Szegő [13] considered the following extremal problem for nonnegativesine polynomials:

Determine the minimum and maximum values of b_k provided that the sine polynomial $S_n(\theta)$ is in

$$\mathcal{S}_n = \left\{ S_n(\theta) = \sin\theta + \sum_{k=2}^{n} b_k \ \sin k\theta \quad : \quad S_n(\theta) \geq 0 \quad \text{for} \quad \theta \in [0, \pi] \right\}.$$

For each of these values, find the extremal sine polynomial which belongs to $\mathcal{S}_n$ and whose coefficient b_k coincides with the corresponding extremal value.

Although Rogosinski and Szegő suggested two methods for obtaining the extrema of the moments b_k, they found the minimal and maximal values only for b_2, b_3, b_{n-1} and b_n. Moreover, in none of the cases the corresponding extremal sine polynomials were determined explicitly.

In this paper we develop a method of solving the above stated problem. The basic tool is a general representation of nonnegativesine polynomials. Note that Fejér and Riesz' theorem implies a representation of nonnegativecosine polynomials (see Lemma 2 below). However, to the best of our knowledge, no similar representation of nonnegativesine polynomial was known. Due to the complicated nature of the problem, we apply the method to find the extremal polynomials, associated with the minima and maxima of b_n, b_{n-1}, and b_{n-2}.

The results containing a variety of nonnegativesine and cosine polynomials are stated in the next section. We formulate and prove only some of the possible consequences of the main results. We do not include, for example, the nonnegativetrigonometric polynomials which are in the convex hulls of the polynomials that we obtain. The remark which follows the statements of Vietoris' [17] results in Askey and Steinig's paper [4] gives an idea how new nonnegativetrigonometric polynomials can be generated by a known sequence of nonnegativetrigonometric polynomials. Our method is developed in Section 3. The extremal problems for b_{n-1} and for b_n are solved in Section 4. In Section 5 the minimal and the maximal values of b_{n-2} are determined. For odd n, the corresponding extremal polynomials are constructed. The limit behaviour of the extremal values of b_{n-2}, for even n, is established. In Section 6 we prove that some of the sequences of cosine polynomials obtained are in fact positive summability kernels. The graphs of some of the nonnegativetrigonometric polynomials constructed in this paper are shown in Section 7.

2. Statement of results.

Theorem 1. *Let $n = 2m+2$ be a even positive integer. Then for every $S_n(\theta) \in \mathcal{S}_n$:*

$$(i) \qquad -\frac{m+1}{m+2} \le b_n \le \frac{m+1}{m+2}.$$

The equality $b_n = (m+1)/(m+2)$ is attained only for the nonnegativesine polynomial

(2.1)

$$\sum_{k=0}^{m} \left(\frac{(m+k+2)(m-k+1)}{(m+1)(m+2)} \sin(2k+1)\theta + \frac{(k+1)^2}{(m+1)(m+2)} \sin(2k+2)\theta \right).$$

The equality $b_n = -(m+1)/(m+2)$ is attained only for the nonnegativesine polynomial

(2.2)

$$\sum_{k=0}^{m} \left(\frac{(m+k+2)(m-k+1)}{(m+1)(m+2)} \sin(2k+1)\theta - \frac{(k+1)^2}{(m+1)(m+2)} \sin(2k+2)\theta \right).$$

$$(ii) \qquad -\frac{m}{m+2} \le b_{n-1} \le 1.$$

The equality $b_{n-1} = 1$ is attained only for the nonnegativesine polynomials

$$\sin\theta + \sum_{k=1}^{m} (2pq \sin 2k\theta + \sin(2k+1)\theta) + pq\sin(2m+2)\theta, \tag{2.3}$$

where the parameters p and q satisfy $p^2 + q^2 = 1$.
The equality $b_{n-1} = -m/(m+2)$ is attained only for the nonnegativesine polynomials

(2.4)

$$\sin\theta + \sum_{k=1}^{m} \left\{ 2pq\left(1 - \frac{2k^2}{m(m+2)}\right)\sin 2k\theta + \left(1 - \frac{2k(k+1)}{m(m+2)}\right)\sin(2k+1)\theta \right\} - pq\sin(2m+2)\theta,$$

where p and q satisfy the relation $p^2 + q^2 = 1$.

An interesting nonnegativesine polynomial is

$$\sum_{k=0}^{m} \frac{(m+k+2)(m-k+1)}{(m+1)(m+2)} \sin(2k+1)\theta,$$

which is a convex combination of (2.1) and (2.2). Observe that the extremal polynomials (2.3) and (2.4) depend on a parameter. Their convex combination also has an intriguing form.

Theorem 2. *Let $n = 2m+1$ be an odd positive integer. Then for every $S_n(\theta) \in \mathcal{S}_n$*

(i) $\quad \dfrac{-m}{m+2} \le b_n \le 1.$

The equality $b_n = 1$ is attained only for the nonnegativesine polynomial

$$\sum_{k=0}^{m} \sin(2k+1)\theta. \tag{2.5}$$

The equality $b_n = -m/(m+2)$ is attained only for

$$\sum_{k=0}^{m} \left((1 - \frac{2k(k+1)}{m(m+2)}) \sin(2k+1)\theta \right). \tag{2.6}$$

(ii) $\quad -1 \le \ b_{n-1} \le 1.$

The equality $b_{n-1} = 1$ is attained only for the nonnegativesine polynomial

$$\sum_{k=1}^{2m} \sin k\theta + \frac{1}{2} \sin(2m+1)\theta. \tag{2.7}$$

The equality $b_{n-1} = -1$ is attained only for the nonnegativesine polynomial

$$\sum_{k=1}^{2m} (-1)^{k+1} \sin k\theta + \frac{1}{2} \sin(2m+1)\theta. \tag{2.8}$$

It is worth noting the extremal property of the sine polynomial (2.5). Another interesting observation is that the polynomial (2.6) coincides with the polynomial (2.4) if $p = 0$ or $q = 0$.

Theorem 3. *Let $n = 2m+1, \ m \ge 2$ be an odd positive integer. Then for every $S_n(\theta) \in \mathcal{S}_n$*

$$\frac{-m+4-\sqrt{m(5m-8)}}{2(m+2)} \le b_{n-2} \le \frac{1+\sqrt{5}}{2}.$$

The equality $b_{n-2} = (1+\sqrt{5})/2$ is attained only for the nonnegativesine polynomial

$$\begin{aligned} \sin\theta \ &+ \ \sum_{k=1}^{m-2} \left(\frac{5+2\sqrt{5}}{5} \right) \sin(2k+1)\theta \\ &+ \ \left(\frac{1+\sqrt{5}}{2} \right) \sin(2m-1)\theta + \left(\frac{5+\sqrt{5}}{10} \right) \sin(2m+1)\theta. \end{aligned} \tag{2.9}$$

The equality $b_{n-2} = (-m+4-\sqrt{m(5m-8)})/(2m+4)$ *is attained only for the nonnegativesine polynomial*

$$
\begin{aligned}
\sin\theta \quad &+ \quad \sum_{k=1}^{m-2}\left(\frac{4+2k+2k^2-m^2}{m^2-4}\right)\sin(2k+1)\theta \\
&- \quad \sum_{k=1}^{m-2}\left(\frac{2(2k(k+1)-2(k^2+k+2)m+m^3)}{(m^2-4)\sqrt{m(5m-8)}}\right)\sin(2k+1)\theta \\
&+ \quad \left(\frac{4-m-\sqrt{m(5m-8)}}{2(m+2)}\right)\sin(2m-1)\theta \\
&- \quad \frac{m}{2m+4}\left(1+\frac{(m-4)}{\sqrt{m(5m-8)}}\right)\sin(2m+1)\theta.
\end{aligned}
\tag{2.10}
$$

In order to formulate our result concerning the extremal values of b_{n-2} for the case where n is even we need the following technical result:

Lemma 1. *For every positive integer* $m \geq 2$ *the cubic polynomial*

$$r_m(y) = -y^3 + \frac{4(m^2+m+2)}{(m+2)^2}y^2 - \frac{4(m^2-m+1)}{(m+2)^2}y + \frac{(m-1)^2}{(m+2)^2}$$

has three positive zeros. Moreover, the largest zero $y(m)$ *of* $r_m(y)$ *does not exceed* $y(\infty) := (3+\sqrt{5})/2$ *and* $y(m) \to y(\infty)$ *as* m *diverges.*

Theorem 4. *Let* $n = 2m+2,\ m \geq 2$ *be an even positive integer and, let* $y(m)$ *be defined as in Lemma 1. Then for every* $S_n(\theta) \in \mathcal{S}_n$*:*

$$|b_{n-2}| \leq \sqrt{y(m)}.$$

In particular,

$$|b_{n-2}| < ((3+\sqrt{5})/2)^{1/2}.$$

Employing a result of Turán [16, Theorem I] we obtain a consequence of Theorem 1, Theorem 2, and Theorem 3 which provides interesting analogues of the classical result of Fejér, Jackson, and Gronwall which states that, for any positive integer n, the sine polynomial

$$\sum_{k=1}^{n}\frac{\sin k\theta}{k} \tag{2.11}$$

is positive in $(0,\pi)$.

Corollary 1. *The sine polynomials*

$$\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{k-1}{n(n+1)}\right)\sin k\theta, \tag{2.12}$$

$$\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{k-1}{n^2-1}\right)\sin k\theta, \tag{2.13}$$

$$\sum_{k=1}^{n}\frac{1}{k}\sin k\theta \quad - \quad \frac{1}{2n}\sin n\theta,$$

$$\sin\theta + \left(\frac{5+2\sqrt{5}}{5}\right)\sum_{k=2}^{n-2}\frac{1}{k}\sin k\theta + \left(\frac{1+\sqrt{5}}{2(n-1)}\right)\sin(n-1)\theta + \left(\frac{5+\sqrt{5}}{10n}\right)\sin n\theta$$

and

$$\begin{aligned}\sin\theta \quad &+ \quad \sum_{k=2}^{n-2}\left(\frac{n^2-2n-2k^2+2k-3}{(n-3)(n+1)k}\right)\sin k\theta \\ &+ \quad \sum_{k=2}^{n-2}\left(\frac{2(n^2-1)(n-3)-4k(n-2)(k-1)}{(n-3)(n+1)k\sqrt{(n-1)(5n-13)}}\right)\sin k\theta \\ &+ \quad \left(\frac{5-n-\sqrt{(n-1)(5n-13)}}{2(n+1)(n-1)}\right)\sin(n-1)\theta \\ &+ \quad \frac{1-n}{2n(n+1)}\left(1+\frac{n-5}{\sqrt{(n-1)(5n-13)}}\right)\sin n\theta\end{aligned}$$

are nonnegative.

As is seen in Figures 1 and 2, the Gibbs phenomenon of the polynomials (2.12) and (2.13) is almost negligible in comparison with the Gibbs phenomenon of the Fejér-Jackson-Gronwall polynomials. Moreover, (2.12) and (2.13) approximate the function $f(\theta) = (\pi-\theta)/2$ uniformly on every compact subset of $(0,\pi]$ much better than (2.11) does.

The method allows us to obtain also some nonnegativecosine polynomials.

Corollary 2. *The following cosine polynomials are nonnegative:*

$$1+2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{2k(n+k+1)}{n(n+2)}\right)\cos k\theta, \tag{2.14}$$

(2.15)

$$\begin{aligned}1+\cos\theta \quad &+ \quad 2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{k}{n+2}\right)\left(1+\frac{k}{2n+3}\right)\cos(2k)\theta \\ &+ \quad \sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1+k\left(\frac{4(k+1)}{2n+3}-\frac{2k+1}{n+2}\right)\right)\cos(2k+1)\theta,\end{aligned}$$

(2.16)

$$\begin{aligned}1-\cos\theta \quad &+ \quad 2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{k}{n+2}\right)\left(1+\frac{k}{2n+3}\right)\cos(2k)\theta \\ &- \quad \sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1+k\left(\frac{4(k+1)}{2n+3}-\frac{2k+1}{n+2}\right)\right)\cos(2k+1)\theta,\end{aligned}$$

$$1+2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{k}{n+2}\right)\left(1+\frac{k}{2n+3}\right)\cos k\theta, \tag{2.17}$$

$$1+2\sum_{k=0}^{n-1}\left(\left(1-\frac{k+1/2}{n+1/2}\right)\cos(2k+1)\theta + \left(1-\frac{k+1}{n+1/2}\right)\cos(2k+2)\theta\right), \tag{2.18}$$

$$1+2\sum_{k=0}^{n-1}\left(\left(-1+\frac{k+1/2}{n+1/2}\right)\cos(2k+1)\theta+\left(1-\frac{k+1}{n+1/2}\right)\cos(2k+2)\theta\right), \tag{2.19}$$

$$1+2\sum_{k=1}^{n}\left(1-\frac{k}{n+1/2}\right)\cos k\theta, \tag{2.20}$$

(2.21)
$$1+\frac{4n+2}{n+1}pq\cos\theta+2\sum_{k=1}^{n}\left\{\left(1-\frac{k}{n+1}\right)\cos 2k\theta+\left(1-\frac{2k-n}{n+1}\right)pq\cos(2k+1)\theta\right\},$$

(2.22)
$$\begin{aligned} 1 \quad &+ \quad \frac{4n-2}{n+1}pq\cos\theta+2\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{2k(n+k+1)}{n(n+2)}\right)\cos 2k\theta \\ &+ \quad 2\sum_{k=1}^{n}\frac{2n^3+3n^2-2n-6kn^2-12kn+4k^3+6k^2+2k}{n(n+1)(n+2)}pq\cos(2k+1)\theta, \end{aligned}$$

where the parameters p *and* q *in* (2.21) *and* (2.22) *satisfy* $p^2+q^2=1$,

(2.23)
$$\frac{3+\sqrt{5}}{4}(m-1)\left(1+2\sum_{k=1}^{m-2}\left(1-\frac{k}{m-1}\right)\cos k\theta\right)+1+(1+\sqrt{5})\sum_{k=1}^{m-2}\cos k\theta+\cos m\theta$$

and

(2.24)
$$\begin{aligned} & \frac{m(-5+2m+\sqrt{m(5m-8)})}{6\sqrt{m(5m-8)}} \\ + \quad & \sum_{k=1}^{m-1}\left(\frac{10k+2k^3-4m-3km^2+m^3}{3m^2-12}\right)\cos k\theta \\ + \quad & \sum_{k=1}^{m-1}\left(\frac{2m^4-6km^3+m^3-8m^2+4k^3m+20km-4m-4k^3+4k}{(3m^2-12)\sqrt{m(5m-8)}}\right)\cos k\theta \\ - \quad & \frac{m}{2(m+2)}\left(1+\frac{m-4}{\sqrt{m(5m-8)}}\right)\cos m\theta. \end{aligned}$$

The polynomial (2.17) is a convex combination of (2.15) and (2.16), and the polynomial (2.20) is a convex combination of (2.18) and (2.19).

The explicit forms of the nonnegativecosine polynomials given by the formulas from (2.14) to (2.20) are somehow very similar to the classical Fejér kernel (1.1). While the coefficients of $F_n(\theta)$ approximate "linearly" the coefficients of the formal Fourier series

$$1+2\sum_{k=1}^{\infty}\cos k\theta \tag{2.25}$$

of Dirac's delta, the coefficients of our cosine polynomials, except for (2.20), provide "quadratic", or even "cubic", approximations to the coefficients of (2.25). This intuitive observation suggests that the above cosine polynomials may be also positive summability kernels. The graphs of (2.14), (2.17), and (2.20) together with the graph of Fejér's kernel are shown in Figures 3 and 4.

In Section 6 we shall prove that the cosine polynomials (2.14), (2.17) or (2.20) are positive summability kernels. Thus we shall establish the L_p, pointwise and almost everywhere convergence of the corresponding convolutions

$$K_n(f;x) = 1/(2\pi)\int_{-\pi}^{\pi} K_n(\theta) f(x-\theta)d\theta.$$

In particular, when the kernel coincides with (2.14), (2.17) or (2.20), the convolutions reduce to the approximating polynomials

$$\begin{aligned} K_{n,1}(f;x) &= \tfrac{a_0(f)}{2} + \sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{2k(n+k+1)}{n(n+2)}\right) \\ &\qquad \times (a_k(f)\cos kx + b_k(f)\sin kx), \\ K_{n,2}(f;x) &= \tfrac{a_0(f)}{2} + \sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\left(1-\frac{k}{n+2}\right)\left(1+\frac{k}{2n+3}\right) \\ &\qquad \times (a_k(f)\cos kx + b_k(f)\sin kx) \end{aligned}$$

and

$$K_{n,3}(f;x) = \frac{a_0(f)}{2} + \sum_{k=1}^{n}\left(1-\frac{k}{n+1/2}\right)(a_k(f)\cos kx + b_k(f)\sin kx),$$

where $a_k(f)$ and $b_k(f)$ denote the Fourier coefficients of f.

Theorem 5. *For any $p, 1 \le p \le \infty$ and for every 2π-periodic function $f \in L_p[-\pi,\pi]$ the sequences $K_{n,j}(f;x),\ j=1,2,3,$ converge to f in $L_p[-\pi,\pi]$.*

Theorem 6. *Let f be a 2π-periodic function, which is integrable in $[-\pi,\pi]$. If, for $x\in[-\pi,\pi]$, the limit $\lim_{h\to 0}(f(x+h)+f(x-h))$ exists, then, for $j=1,2,3,$*

$$K_{n,j}(f;x) \longrightarrow (1/2)\lim_{h\to 0}(f(x+h)+f(x-h)) \quad \text{as} \quad n \quad \text{diverges.}$$

Theorem 7. *Let f be a 2π-periodic function, which is integrable in $[-\pi,\pi]$. Then $K_{n,j}(f;x),\ j=1,2,3,$ converge to f almost everywhere in $[-\pi,\pi]$.*

It is worth mentioning that all the cosine polynomials we have constructed, some with slight modification, are positive summability kernels. More precisely, (2.23) and (2.24) are also positive summability kernels. In the remaining four cases, namely when $C_n(\theta)$ is one of the cosine polynomials (2.15), (2.16), (2.18) or (2.19), then $K_n(\theta) = C_n(\theta/2)$ is a positive summability kernel. We shall omit the proofs of these technical results.

3. Fejér-Riesz type representation of nonnegativesine polynomials

A careful inspection of the proof of the Fejér-Riesz theorem shows that the coefficients of any nonnegativecosine polynomial are representable in terms of real parameters c_k.

Lemma 2. *Let*

$$C_n(\theta) = a_0 + 2\sum_{k=1}^{n} a_k \cos k\theta$$

be a cosine polynomial of order n which is nonnegativefor every real θ. Then there exists an algebraic polynomial with real coefficients $R(z) = \sum_{k=0}^{n} c_k z^k$ of degree n such that $C_n(\theta) = |R(e^{i\theta})|^2$. Thus, the cosine polynomial $C_n(\theta)$ of order n is nonnegativeif and only if there exist real numbers c_k, $k = 0, 1, \ldots, n$, such that

$$\begin{aligned} a_0 &= \sum_{k=0}^{n} c_k^2, \\ a_k &= \sum_{\nu=0}^{n-k} c_{k+\nu} c_\nu \quad \text{for} \quad k = 1, \ldots, n. \end{aligned} \tag{3.1}$$

The following relation between nonnegativesine and nonnegativecosine polynomials is a simple observation (see [13]).

Lemma 3. *The sine polynomial of order n*

$$S_n(\theta) = \sum_{k=1}^{n} b_k \sin k\theta$$

is nonnegativein $[0, \pi]$ if and only if the cosine polynomial of order $n-1$:

$$C_{n-1}(\theta) = a_0 + 2\sum_{k=1}^{n-1} a_k \cos k\theta,$$

where

$$\begin{aligned} b_k &= a_{k-1} - a_{k+1} \quad \text{for} \quad k = 1, \ldots, n-2, \\ b_{n-1} &= a_{n-2}, \\ b_n &= a_{n-1}, \end{aligned} \tag{3.2}$$

is nonnegative.

As a consequence of the last two lemmas we obtain a parametric representation for the coefficients of the nonnegativesine polynomials.

Lemma 4. *The sine polynomial of order n:*

$$S_n(\theta) = \sum_{k=1}^{n} b_k \sin k\theta$$

is nonnegativeif and only if there exist real numbers $c_0, \ldots, c_{n-1}$*, such that*

$$
\begin{aligned}
b_1 &= \sum_{\nu=0}^{n-1} c_\nu^2 - \sum_{\nu=0}^{n-3} c_\nu c_{\nu+2}, \\
b_k &= \sum_{\nu=0}^{n-k} c_{k+\nu-1} c_\nu - \sum_{\nu=0}^{n-k-2} c_{k+\nu+1} c_\nu, \quad \text{for} \quad k = 2, \ldots, n-2, \\
b_{n-1} &= c_0 c_{n-2} + c_1 c_{n-1}, \\
b_n &= c_0 c_{n-1}.
\end{aligned}
\tag{3.3}
$$

The principal idea is to canonize the quadratic form, which represents b_1 in order to obtain sum of squares. The process of canonization begins as follows:

$$
\begin{aligned}
b_1 &= \sum_{k=0}^{n-1} c_k^2 - \sum_{k=0}^{n-3} c_k c_{k+2} \\
&= \left(c_0 - \frac{1}{2} c_2\right)^2 + \left(c_1 - \frac{1}{2} c_3\right)^2 + \frac{3}{4} c_2^2 + \frac{3}{4} c_3^2 + \sum_{k=4}^{n-1} c_k^2 - \sum_{k=2}^{n-3} c_k c_{k+2} \\
&= \left(c_0 - \frac{1}{2} c_2\right)^2 + \left(c_1 - \frac{1}{2} c_3\right)^2 + \left(\frac{\sqrt{3}}{2} c_2 - \frac{1}{\sqrt{3}} c_4\right)^2 + \left(\frac{\sqrt{3}}{2} c_3 - \frac{1}{\sqrt{3}} c_5\right)^2 \\
&\quad + \frac{2}{3} c_4^2 + \frac{2}{3} c_5^2 + \sum_{k=6}^{n-1} c_k^2 - \sum_{k=4}^{n-3} c_k c_{k+2}.
\end{aligned}
$$

It is not difficult to see that if n is odd, $n = 2m+1$, then b_1 can be written as a sum of squares in the form

$$
\begin{aligned}
b_1 &= \sum_{k=0}^{m-2} \left\{ \left(\sqrt{q_{k+1}} c_{2k} - \frac{c_{2k+2}}{2\sqrt{q_{k+1}}} \right)^2 + \left(\sqrt{q_{k+1}} c_{2k+1} - \frac{c_{2k+3}}{2\sqrt{q_{k+1}}} \right)^2 \right\} \\
&\quad + \left(\sqrt{q_m} c_{2m-2} - \frac{c_{2m}}{2\sqrt{q_m}} \right)^2 + q_m c_{2m-1}^2 + q_{m+1} c_{2m}^2,
\end{aligned}
$$

and if n is even, $n = 2m+2$, then b_1 is given by

$$
\begin{aligned}
b_1 &= \sum_{k=0}^{m-1} \left\{ \left(\sqrt{q_{k+1}} c_{2k} - \frac{c_{2k+2}}{2\sqrt{q_{k+1}}} \right)^2 + \left(\sqrt{q_{k+1}} c_{2k+1} - \frac{c_{2k+3}}{2\sqrt{q_{k+1}}} \right)^2 \right\} \\
&\quad + q_{m+1} c_{2m}^2 + q_{m+1} c_{2m+1}^2,
\end{aligned}
$$

where the parameters $q_k,\ k = 1, \ldots, m+1$, are determined by the recurrence relation $q_1 = 1,\ q_{k+1} = 1 - \ 1/(4q_k)$. The solution of this recurrence equation is

$$
q_k = \frac{k+1}{2k}. \tag{3.4}
$$

We can formulate our next technical lemma:

Lemma 5. *Let the sequence* $\{q_k\}$ *be defined by* (3.4). *Then, for every positive integer* n*, the coefficient* b_1 *can be represented as a sum of squares*

$$
b_1 = \sum_{k=0}^{n-1} d_k^2,
$$

where, for n odd, $n = 2m + 1$, the parameters d_k and c_k are related by

$$\begin{aligned}
\sqrt{q_k}c_{2k-2} - \frac{1}{2}\frac{1}{\sqrt{q_k}}c_{2k} &= d_{2k-2}, \quad k = 1, \dots, m,\\
\sqrt{q_k}c_{2k-1} - \frac{1}{2}\frac{1}{\sqrt{q_k}}c_{2k+1} &= d_{2k-1}, \quad k = 1, \dots, m-1,\\
\sqrt{q_{m+1}}c_{2m} &= d_{2m},\\
\sqrt{q_m}c_{2m-1} &= d_{2m-1},
\end{aligned}$$

and for n even, $n = 2m + 2$, d_k and c_k are related by

$$\begin{aligned}
\sqrt{q_k}c_{2k-2} - \frac{1}{2}\frac{1}{\sqrt{q_k}}c_{2k} &= d_{2k-2}, \quad k = 1, \dots, m,\\
\sqrt{q_k}c_{2k-1} - \frac{1}{2}\frac{1}{\sqrt{q_k}}c_{2k+1} &= d_{2k-1}, \quad k = 1, \dots, m,\\
\sqrt{q_{m+1}}c_{2m} &= d_{2m},\\
\sqrt{q_{m+1}}c_{2m+1} &= d_{2m+1}.
\end{aligned}$$

Solve the above systems of linear equations with respect to c_k. Bearing in mind the explicit form (3.4) of q_k, $k = 1, \dots$, and setting $q_{k+1} \cdots q_{k+\nu} = 1$ for $\nu = 0$, we obtain an explicit representation of c_k in terms of a linear combination of the new parameters d_j, $j = k, k+2, k+4, \dots$. Thus c_{2k}, $k = 0, 1, \dots, m$, are given by

$$\begin{aligned}
c_{2k} &= \sum_{\nu=0}^{m-k} \frac{1}{2^\nu}\left(q_{k+1} \cdots q_{k+\nu}\sqrt{q_{k+\nu+1}}\right)^{-1} d_{2k+2\nu}\\
&= (k+1)\sum_{\nu=0}^{m-k} \sqrt{\frac{2}{(k+\nu+1)(k+\nu+2)}}\, d_{2k+2\nu},
\end{aligned} \tag{3.5}$$

while for the parameters c_{2k+1} we have

$$\begin{aligned}
c_{2k+1} &= \sum_{\nu=0}^{m-k-1} \frac{1}{2^\nu}\left(q_{k+1} \cdots q_{k+\nu}\sqrt{q_{k+\nu+1}}\right)^{-1} d_{2k+2\nu+1}\\
&= (k+1)\sum_{\nu=0}^{m-k-1} \sqrt{\frac{2}{(k+\nu+1)(k+\nu+2)}}\, d_{2k+2\nu+1},
\end{aligned} \tag{3.6}$$

if $n = 2m + 1$, or

$$\begin{aligned}
c_{2k+1} &= \sum_{\nu=0}^{m-k} \frac{1}{2^\nu}\left(q_{k+1} \cdots q_{k+\nu}\sqrt{q_{k+\nu+1}}\right)^{-1} d_{2k+2\nu+1}\\
&= (k+1)\sum_{\nu=0}^{m-k} \sqrt{\frac{2}{(k+\nu+1)(k+\nu+2)}}\, d_{2k+2\nu+1},
\end{aligned} \tag{3.7}$$

if $n = 2m + 2$.

Lemma 4, Lemma 5, and relations (3.5), (3.6), and (3.7) yield the desired parametric representation of the coefficients of nonnegativesine polynomials.

Theorem 8. *The sine polynomial*

$$S_n(\theta) = \sum_{k=1}^{n} b_k \sin k\theta$$

is nonnegativefor $\theta \in [0,\pi]$ if and only if there exist real parameters $d_0,\dots,d_{n-1}$, such that $b_1 = d_0^2 + \cdots + d_{n-1}^2$ and

$$b_k = \sum_{i=0}^{n-1} \delta_{i,i}^{(k)} d_i^2 + 2 \sum_{0\le i<j\le n-1} \delta_{i,j}^{(k)} d_i d_j \quad \text{for} \quad k = 1,\dots,n-1, \tag{3.8}$$

where the coefficients $\delta_{i,j}^{(k)}$ are explicitly obtained by the relations (3.3)*,* (3.5)*,* (3.6) *and* (3.7)*.*

Let $D_k = (\delta_{i,j}^{(k)})_{i,j=0}^{n-1}$ be a symmetric matrix associated with the quadratic form (3.8). Denote by $\mathbf{d}$ the vector $(d_0,\dots,d_{n-1})^T$. Then

$$b_k = \mathbf{d}^T D_k \mathbf{d}.$$

By the Rayleigh-Ritz theorem (see Theorem 4.2.2 on p. 176 in Horn and Johnson [9]) we obtain the main result in this section.

Theorem 9. *Let $\lambda_{k,min}$ and $\lambda_{k,max}$ be the smallest and the largest eigenvalues of D_k. If $S_n(\theta) \in \mathcal{S}_n$, then*

$$\lambda_{k,min} \le b_k \le \lambda_{k,max}.$$

Moreover, the equality $b_k = \lambda_{k,max}$ is attained if and only if $S_n(\theta)$ is a sine polynomial in $\mathcal{S}_n$ whose coefficients b_i, $i = 2,\dots,n$, are obtained by the formulas

$$b_i = \mathbf{d}_{k,max}^T D_i \mathbf{d}_{k,max},$$

where $\mathbf{d}_{k,max}$ is an eigenvector with unit length, associated with the eigenvalue $\lambda_{k,max}$.

Similarly, the equality $b_k = \lambda_{k,min}$ is attained if and only if $S_n(\theta)$ is a sine polynomial in $\mathcal{S}_n$, whose coefficients b_i, $i = 2,\dots,n$, are obtained by the formulas

$$b_i = \mathbf{d}_{k,min}^T D_i \mathbf{d}_{k,min},$$

where $\mathbf{d}_{k,min}$ is an eigenvector with unit length, associated with the eigenvalue $\lambda_{k,min}$.

4. Extremal polynomials for b_n and b_{n-1}

In this section we find the extremal values and the corresponding extremal polynomials for the coefficients b_n and b_{n-1}. It is worth mentioning that the extremal values as well as the associated extremal polynomials depend on the parity of n. Thus, for any coefficient we consider four cases: the problems of determining the minimum and the maximum and, for each of these, the cases of even and odd n.

4.1. The minimum and maximum of b_n for $n = 2m+1$ and the extremal polynomials associated.

On using the relations (3.3), (3.5) and (3.6) we obtain

$$b_n = \sum_{k=0}^{m} \frac{d_{2k} d_{2m}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_{m+1}}},$$

where q_k is given by (3.4). Since

$$2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_{m+1}} = \sqrt{\frac{m+2}{m+1}} \frac{\sqrt{(k+1)(k+2)}}{2},$$

then

$$b_n = \sqrt{\frac{m+1}{m+2}} \sum_{k=0}^{m} \frac{2d_{2k}d_{2m}}{\sqrt{(k+1)(k+2)}}.$$

Set

$$\alpha := \sqrt{\frac{m+1}{m+2}} \tag{4.1}$$

and

$$A_j := \frac{1}{\sqrt{(j+1)(j+2)}}, \quad j = 0, \dots, m. \tag{4.2}$$

Then the symmetric matrix $D_n^{(o)}$ associated with the quadratic form which represents b_n is

$$D_n^{(o)} = \begin{pmatrix} 2\alpha A_m & 0 & \alpha A_{m-1} & \cdots & \alpha A_1 & 0 & \alpha A_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha A_{m-1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \\ \alpha A_1 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ \alpha A_0 & 0 & 0 & & & & \end{pmatrix}.$$

Hence the characteristic polynomial of $D_n^{(o)}$ is

$$\begin{aligned} det(D_n^{(o)} - \lambda I) &= (-\lambda)^{2m} \left\{(\alpha^2 A_0^2 + \cdots + \alpha^2 A_{m-1}^2)/\lambda + 2\alpha A_m - \lambda\right\} \\ &= (-\lambda)^{2m-1} \left\{\lambda^2 - 2\frac{\alpha}{\sqrt{(m+1)(m+2)}}\lambda - \alpha^2 \frac{m}{m+1}\right\}. \end{aligned}$$

Since the smallest and the largest zeros of this polynomial are $-m/(m+2)$ and 1, then, by Theorem 9,

$$-\frac{m}{m+2} \le b_n \le 1.$$

In order to determine the extremal polynomial associated with $b_{n,max}$ we need to find first an eigenvector $\mathbf{d}_{n,max}^{(o)}$ of $D_n^{(o)}$, associated with the eigenvalue $\lambda = 1$, such that

$$d_0^2 + \cdots + d_{2m}^2 = 1. \tag{4.3}$$

From the homogeneous linear system

$$\begin{pmatrix} 2\alpha A_m - 1 & 0 & \alpha A_{m-1} & \cdots & \alpha A_1 & 0 & \alpha A_0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \alpha A_{m-1} & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \vdots \\ \alpha A_1 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ \alpha A_0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} d_{2m} \\ d_{2m-1} \\ d_{2m-2} \\ \vdots \\ d_2 \\ d_1 \\ d_0 \end{pmatrix} = 0$$

for the coordinates of $\mathbf{d}_{n,max}^{(o)}$, we obtain $d_{2k} = \alpha A_k d_{2m}$ and $d_{2k+1} = 0$ for $k = 0, \dots, m-1$. A substitution of these values of d_j in (4.3) yields $d_{2m} = \pm((m+$

$2)/(2m+2))^{1/2}$. Thus the desired eigenvectors are $\pm\mathbf{d}^{(o)}_{n,max}$, where

$$\mathbf{d}^{(o)}_{n,max} = \frac{1}{\sqrt{2}}\left(A_0, 0, A_1, 0, \ldots, A_{m-2}, 0, A_{m-1}, 0, 1/\alpha\right)^T.$$

Observe that both the coefficients a_k and b_k are expressed as quadratic forms of $d_0, d_1, \ldots, d_{2m}$. Thus any choice of the sign of the eigenvector will imply the same extremal polynomial. We prefer working with $\mathbf{d}^{(o)}_{n,max}$. From (3.5) and (3.6) we obtain $c_{2k+1} = 0$ for $k = 0, \ldots, m-1$, and

$$\begin{aligned} c_{2k} &= (k+1)\sum_{\nu=0}^{m-k}\sqrt{\frac{2}{(k+\nu+1)(k+\nu+2)}}d_{2k+2\nu} \\ &= \frac{k+1}{m+1} + (k+1)\sum_{\nu=0}^{m-k-1}\frac{1}{(k+\nu+1)(k+\nu+2)} \\ &= 1. \end{aligned}$$

Then the expressions (3.1) yield $a_{2k} = m+1-k$ for $k = 0, \ldots, m$, and $a_{2k+1} = 0$ for $k = 0, \ldots, m-1$. Thus we obtained the classical nonnegativecosine polynomial (1.1). On using (3.2) we obtain (2.5).

The procedure of determining an extremal sine polynomial associated with $b_{n,min}$ is similar. First we find the eigenvectors of $D^{(o)}_n$, associated with the smallest eigenvalue $\lambda = -m/(m+2)$. They are $\pm\mathbf{d}_{n,min}$, where

$$\mathbf{d}_{n,min} = \sqrt{\frac{m+2}{2m}}\left(A_0, 0, A_1, 0, \ldots, 0, A_{m-1}, 0, -mA_m\right)^T,$$

and we choose the positive sign. Thus, $d_{2k+1} = 0$, $d_{2m} = -\sqrt{m/(2m+2)}$, and

$$d_{2k+2\nu} = \sqrt{\frac{m+2}{2m}}\frac{1}{(k+\nu+1)(k+\nu+2)} \quad \text{for} \quad k+\nu = 0, \ldots, m-1.$$

Then, by (3.6) we obtain $c_{2k+1} = 0$, and by (3.5) we get

$$\begin{aligned} c_{2k} &= (k+1)\sum_{\nu=0}^{m-k}\sqrt{\frac{2}{(k+\nu+1)(k+\nu+2)}}d_{2k+2\nu} \\ &= -\frac{k+1}{m+1}\sqrt{\frac{m}{m+2}} + (k+1)\sqrt{\frac{m}{m+2}}\sum_{\nu=0}^{m-k-1}\frac{1}{(k+\nu+1)(k+\nu+2)} \\ &= \sqrt{\frac{m}{m+2}}\frac{m-2k}{m} \quad \text{for} \quad k = 0, \ldots, m. \end{aligned}$$

Hence $a_{2k+1} = 0$ and the general formula for a_{2k} in terms of $c_0, c_2, \ldots, c_{2m}$ yields

$$\begin{aligned} a_{2k} &= \frac{1}{m(m+2)}\sum_{j=0}^{m-k}(m-2j)(m-2j-2k) \\ &= \frac{(m-k+1)(m^2+2m-2mk-2k^2-2k)}{3m(m+2)}. \end{aligned}$$

We arrived at the nonnegativecosine polynomial (2.14). Now we need to express the coefficients b_k in terms of a_k by (3.2). Obviously, $b_{2k} = 0$ and

$$b_{2k+1} = 1 - \frac{2k(k+1)}{m(m+2)}.$$

This gives the nonnegativesine polynomial (2.6).

4.2. **The minimum and maximum of b_n for $n = 2m + 2$ and the extremal polynomials associated.**

Relations (3.3), (3.5), and (3.7) imply

$$b_n = \sum_{k=0}^{m} \frac{d_{2k} d_{2m+1}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}} \sqrt{q_{m+1}}}.$$

Since

$$2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}} \sqrt{q_{m+1}} = \sqrt{\frac{m+2}{m+1}} \frac{\sqrt{(k+1)(k+2)}}{2},$$

then

$$b_n = \sqrt{\frac{m+1}{m+2}} \sum_{k=0}^{m} \frac{2 d_{2k} d_{2m+1}}{\sqrt{(k+1)(k+2)}}.$$

Hence the symmetric matrix $D_n^{(e)}$ associated with the quadratic form which represents b_n is

$$D_n^{(e)} = \begin{pmatrix} 0 & \alpha A_m & 0 & \cdots & \alpha A_1 & 0 & \alpha A_0 \\ \alpha A_m & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \\ \alpha A_1 & 0 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ \alpha A_0 & 0 & 0 & & & & \end{pmatrix},$$

where α and A_j are defined by (4.1) and (4.2). The characteristic polynomial of $D_n^{(e)}$ is

$$\begin{aligned} det(D_n^{(e)} - \lambda I) &= (-\lambda)^{2m+1} \left\{ (\alpha^2 A_0^2 + \cdots + \alpha^2 A_{m-1}^2 + \alpha^2 A_m^2)/\lambda - \lambda \right\} \\ &= (-\lambda)^{2m} \left\{ \lambda^2 - \alpha^2 \frac{m+1}{m+2} \right\}. \end{aligned}$$

Since the smallest and the largest zeros of this polynomial are $-(m+1)/(m+2)$ and $(m+1)/(m+2)$, then

$$-\frac{m+1}{m+2} \le b_n \le \frac{m+1}{m+2}.$$

First we obtain an eigenvector $\mathbf{d}_{n,max}^{(e)}$ of $D_n^{(e)}$, associated with the eigenvalue $\lambda = (m+1)/(m+2)$, such that

$$d_0^2 + \cdots + d_{2m+1}^2 = 1. \tag{4.4}$$

Solve the homogeneous linear system $(D_n^{(e)} - ((m+1)/(m+2))I)\mathbf{d}_{n,max}^{(e)} = 0$ we obtain $d_{2k} = ((m+2)/(m+1))\alpha A_k d_{2m+1}$, $k = 0, \ldots, m$, and $d_{2k+1} = 0$, $k = 0, \ldots, m-1$. Substitute these values of d_j in (4.4) to get $d_{2m+1} = \pm\sqrt{2}/2$. Thus the eigenvectors we need are $\pm\mathbf{d}_{n,max}^{(e)}$, where

$$\mathbf{d}_{n,max}^{(e)} = \frac{1}{\sqrt{2}\alpha} \left(A_0, 0, A_1, 0, \ldots, A_m, \alpha \right)^T.$$

We shall work with $\mathbf{d}_{n,max}^{(e)}$. By (3.5) and (3.7), we obtain

$$c_{2k} = \frac{m-k+1}{m+2}\sqrt{\frac{m+2}{m+1}} \quad \text{for} \quad k = 0, \dots, m,$$

and

$$c_{2k+1} = \frac{k+1}{m+1}\sqrt{\frac{m+1}{m+2}} \quad \text{for} \quad k = 0, \dots, m.$$

Then formulas (3.1) imply that

$$a_{2k} = \frac{(m-k+1)(2m^2 - mk + 7m - k^2 - k + 6)}{3(m+1)(m+2)}$$

and

$$a_{2k+1} = \frac{(2m^3 + 9m^2 + 13m - 2k^3 - 3k^2 - k + 6)}{6(m+1)(m+2)}.$$

Thus we obtained the nonnegativecosine polynomial (2.15).On using relations (3.2) we obtain the nonnegative sine polynomial (2.1).

In order to determine the extremal sine polynomial associated with $b_{n,min}$, we find the eigenvectors of $D_n^{(e)}$, associated with the smallest eigenvalue $\lambda = -(m+1)/(m+2)$. They are $\pm\mathbf{d}_{n,min}^{(e)}$, where

$$\mathbf{d}_{n,min}^{(e)} = -\frac{1}{\sqrt{2}\alpha}\left(A_0, 0, A_1, 0, \dots, A_m, -\alpha\right)^T,$$

and we choose the positive sign. Then formulas (3.5) and (3.7) yield

$$c_{2k} = -\frac{m-k+1}{m+2}\sqrt{\frac{m+2}{m+1}} \quad \text{and} \quad c_{2k+1} = \frac{k+1}{m+1}\sqrt{\frac{m+1}{m+2}}$$

for $k = 0, \dots, m$. Then,

$$a_{2k} = \frac{(m-k+1)(2m^2 - mk + 7m - k^2 - k + 6)}{3(m+1)(m+2)}$$

and

$$a_{2k+1} = -\frac{(2m^3 + 9m^2 + 13m - 2k^3 - 3k^2 - k + 6)}{6(m+1)(m+2)}.$$

Thus we obtain the nonnegativecosine polynomial (2.16). On using the relations (3.2) we obtain the nonnegative sine polynomial (2.2).

4.3. **The minimum and maximum of b_{n-1} for $n = 2m+1$ and the extremal polynomials associated.**

In this case

$$\begin{aligned} b_{n-1} &= \sum_{k=0}^{m} \frac{d_{2k}d_{2m-1}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_m}} + \sum_{k=0}^{m-1} \frac{d_{2k+1}d_{2m}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_{m+1}}} \\ &= \sqrt{\frac{m}{m+1}} \sum_{k=0}^{m} \frac{2d_{2k}d_{2m-1}}{\sqrt{(k+1)(k+2)}} + \sqrt{\frac{m+1}{m+2}} \sum_{k=0}^{m-1} \frac{2d_{2k+1}d_{2m}}{\sqrt{(k+1)(k+2)}}. \end{aligned}$$

Setting

$$\beta := \sqrt{\frac{m}{m+1}}, \tag{4.5}$$

we obtain the symmetric matrix

$$D^{(o)}_{n-1} = \begin{pmatrix} 0 & \beta A_m + \alpha A_{m-1} & 0 & \cdots & \alpha A_0 & 0 \\ \beta A_m + \alpha A_{m-1} & 0 & \beta A_{m-1} & \cdots & 0 & \beta A_0 \\ 0 & \beta A_{m-1} & & & & \\ \alpha A_{m-2} & 0 & & & & \\ \vdots & \vdots & & & & \\ 0 & \beta A_1 & & & & \\ \alpha A_0 & 0 & & & & \\ 0 & \beta A_0 & & & & \end{pmatrix}$$

associated with the quadratic form which represents b_{n-1}. Hence the characteristic polynomial of $D^{(o)}_{n-1}$ is

$$det(D^{(o)}_{n-1} - \lambda I) = (-\lambda)^{2m-3} \left\{ \lambda^4 - \frac{2m^2+2m+2}{(m+1)(m+2)} \lambda^2 + \frac{m(m-1)}{(m+1)(m+2)} \right\},$$

whose smallest and largest zeros are ± 1. By Theorem 9, we have

$$|b_{n-1}| \leq 1.$$

In order to determine the extremal polynomial associated with $b_{n-1,max}$ we need to find an eigenvector $\mathbf{d}^{(o)}_{n-1,max}$ of $D^{(o)}_{n-1}$, associated with the eigenvalue $\lambda = 1$, such that $d_0^2 + \cdots + d_{2m}^2 = 1$. For the eigenvector $\mathbf{d}^{(o)}_{n-1,max}$, we obtain

$$\mathbf{d}^{(o)}_{n-1,max} = \frac{1}{2} \left(A_0, A_0, \ldots, A_{m-2}, A_{m-2}, A_{m-1}, \sqrt{\frac{m+1}{m}}, \sqrt{\frac{m+2}{m+1}} \right)^T.$$

The relations (3.5) and (3.6) yield $c_k = \sqrt{2}/2$ for $k = 0, \ldots, 2m$. Then the expressions (3.1) imply $a_{2k} = m - k + 1/2$ for $k = 0, \ldots, m$ and $a_{2k+1} = m - k$ for $k = 0, \ldots, m-1$. We obtain the nonnegativecosine polynomial (2.18) and, by (3.2), the positive sine polynomial (2.7).

The extremal sine polynomial associated with $b_{n-1,min}$ is obtained through the eigenvectors of $D^{(o)}_{n-1}$, associated with the smallest eigenvalue $\lambda = -1$. They are $\pm\mathbf{d}^{(e)}_{n,min}$, where

$$\mathbf{d}^{(o)}_{n-1,min} = \frac{1}{2} \left(A_0, -A_0, \ldots, A_{m-2}, -A_{m-2}, A_{m-1}, -\sqrt{\frac{m+1}{m}}, \sqrt{\frac{m+2}{m+1}} \right)^T.$$

The relations (3.5) and (3.6) yield $c_{2k} = \sqrt{2}/2$ for $k = 0, \ldots, m$ and $c_{2k+1} = -\sqrt{2}/2$ for $k = 0, \ldots, m-1$. Then the expressions (3.1) imply $a_{2k} = m - k + 1/2$ for $k = 0, \ldots, m$ and $a_{2k+1} = -(m-k)$ for $k = 0, \ldots, m-1$. Thus we obtain the nonnegativecosine polynomial (2.19). Then, by (3.2), we obtain the nonnegative sine polynomial (2.8).

4.4. **The minimum and maximum of b_{n-1} for $n = 2m+2$ and the extremal polynomials associated.**

Since in this case

$$\begin{aligned} b_{n-1} &= \sum_{k=0}^{m}\left(\frac{d_{2k}d_{2m}}{2^k q_1 q_2\cdots q_k\sqrt{q_{k+1}}\sqrt{q_{m+1}}}+\frac{d_{2k+1}d_{2m+1}}{2^k q_1 q_2\cdots q_k\sqrt{q_{k+1}}\sqrt{q_{m+1}}}\right)\\ &= \sqrt{\frac{m+1}{m+2}}\sum_{k=0}^{m}\frac{2(d_{2k}d_{2m}+d_{2k+1}d_{2m+1})}{\sqrt{(k+1)(k+2)}}, \end{aligned}$$

then the symmetric matrix $D_{n-1}^{(e)}$ associated with the quadratic form which represents b_{n-1} is

$$D_{n-1}^{(e)} = \begin{pmatrix} 2\alpha A_m & 0 & \alpha A_{m-1} & 0 & \cdots & \alpha A_0 & 0\\ 0 & 2\alpha A_m & 0 & \alpha A_{m-1} & \cdots & 0 & \alpha A_0\\ \alpha A_{m-1} & 0 & & & & & \\ 0 & \alpha A_{m-1} & & & & & \\ \vdots & \vdots & & & & & \\ \alpha A_0 & 0 & & & & & \\ 0 & \alpha A_0 & & & & & \end{pmatrix}.$$

Hence the characteristic polynomial of $D_{n-1}^{(e)}$ is

$$det(D_{n-1}^{(e)}-\lambda I) = \lambda^{2m-2}\left(\lambda^2-\frac{2}{m+2}\lambda-\frac{m}{m+2}\right)^2,$$

whose smallest and largest zeros are $-m/(m+2)$ and 1. Therefore

$$-\frac{m}{m+2}\le b_{n-1}\le 1.$$

In order to determine the extremal polynomial associated with $b_{n-1,max}$ we need to find first an eigenvector $\mathbf{d}_{n-1,max}^{(e)}$ of $D_{n-1}^{(e)}$, associated with the eigenvalue $\lambda=1$, such that

$$d_0^2+\cdots+d_{2m+1}^2=1.$$

As in the previous cases, for the coordinates of the eigenvector $\mathbf{d}_{n-1,max}^{(o)}$, we obtain $d_{2m}=P$, $d_{2m+1}=Q$, and $d_{2k}=\alpha PA_k$ and $d_{2k+1}=\alpha QA_k$ for $k=0,\dots,m-1$, where the parameters P and Q satisfy $P^2+Q^2=(m+2)/(2m+2)$. Now the relations (3.5) and (3.7) yield $c_{2k}=\sqrt{2}\alpha P$ and $c_{2k+1}=\sqrt{2}\alpha Q$ for $k=0,\dots,m$. Then the expressions (3.1) imply

$$a_{2k}=m-k+1 \quad\text{and}\quad a_{2k+1}=\frac{2(m+1)(2m-2k+1)}{m+2}PQ \quad\text{for}\quad k=0,\dots,m.$$

On substituting

$$P=\sqrt{\frac{m+2}{2m+2}}p \quad\text{and}\quad Q=\sqrt{\frac{m+2}{2m+2}}q$$

we obtain the nonnegativecosine polynomial (2.21) and the relations (3.2) yield the nonnegative sine polynomial (2.3).

The eigenvector $\mathbf{d}_{n-1,min}^{(e)}$ of $D_{n-1}^{(e)}$, associated with the smallest eigenvalue $\lambda=-m/(m+2)$ is given by $d_{2m}=P$, $d_{2m+1}=Q$, and $d_{2k}=-\alpha PA_k(m+2)/m$ and $d_{2k+1}=\alpha QA_k(m+2)/m$ for $k=0,\dots,m-1$, where in this case the parameters P and Q satisfy $P^2+Q^2=m/(2m+2)$. Now the relations (3.5) and (3.7) yield

$c_{2k} = \sqrt{2}\alpha P(2k-m)/m$ and $c_{2k+1} = \sqrt{2}\alpha Q(2k-m)/m$ for $k = 0, \dots, m$. By (3.1) we obtain

$$a_{2k} = \frac{(m-k+1)(m^2-2km+2m-2k^2-2k)}{3m(m+2)} \quad \text{for} \quad k = 0, \dots, m,$$

and

$$a_{2k+1} = \frac{2(m+1)(4k^3+6k^2+2k-2m-12km+3m^3-6km^2+2m^3)}{3m^2(m+2)} PQ$$

for $k = 0, \dots, m$. Substituting

$$P = \sqrt{\frac{m}{2m+2}p} \quad \text{and} \quad Q = \sqrt{\frac{m}{2m+2}q}$$

we obtain the nonnegativecosine polynomial (2.22) and then, by (3.2), we obtain the nonnegative sine polynomial (2.4).

5. Extremal problems for b_{n-2}

5.1. The minimum and maximum of b_{n-2} for $n = 2m+1$ and the extremal polynomials associated.

Proof of Theorem 3. In this case (3.3), (3.5) and (3.6) imply

$$\begin{aligned} b_{n-2} &= \sum_{k=0}^{m} \left(\frac{d_{2k}d_{2m-2}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_m}} + \frac{d_{2k}d_{2m}}{2^{k+1} q_1 q_2 \cdots q_k q_m \sqrt{q_{k+1}}\sqrt{q_{m+1}}} \right) \\ &\quad + \sum_{k=0}^{m} \frac{(-1)^{2^k} d_{2k}d_{2m}}{2^k q_1 q_2 \cdots q_k q_m \sqrt{q_{k+1}}\sqrt{q_{m+1}}} + \sum_{k=0}^{m-1} \frac{d_{2k+1}d_{2m-1}}{2^k q_1 q_2 \cdots q_k \sqrt{q_{k+1}}\sqrt{q_m}} \\ &= \sum_{k=0}^{m} \frac{2}{\sqrt{(k+1)(k+2)}} \left(\beta d_{2k}d_{2m-2} + \gamma d_{2k}d_{2m} + (-1)^{2^k} \alpha d_{2k}d_{2m} \right) \\ &\quad + \sum_{k=0}^{m-1} \frac{2\beta d_{2k+1}d_{2m-1}}{\sqrt{(k+1)(k+2)}}, \end{aligned}$$

where α and β are defined by (4.1) and (4.5) and

$$\gamma := \frac{m}{\sqrt{(m+1)(m+2)}}. \tag{5.1}$$

Thus, the symmetric matrix $D^{(o)}_{n-2}$ associated with the quadratic form which represents b_{n-2} is

$$\begin{pmatrix} 2(\alpha+\gamma)\alpha A_m & 0 & (\alpha+\gamma)A_{m-1} + \alpha A_m & \cdots & (\gamma-\alpha)A_0 \\ 0 & 2\beta A_{m-1} & 0 & \cdots & 0 \\ (\alpha+\gamma)A_{m-1} + \alpha A_m & 0 & 2\beta A_{m-1} & \cdots & \beta A_0 \\ \vdots & \vdots & \vdots & & \\ (\alpha+\gamma)A_1 & 0 & \beta A_1 & & \\ 0 & \beta A_0 & 0 & & \\ (\gamma-\alpha)A_0 & 0 & \beta A_0 & & \end{pmatrix}.$$

Then, by straightforward calculations, we obtain the characteristic polynomial of $D^{(o)}_{n-2}$ in the form

$$-\lambda^{2m-5}\left(\lambda-1\right)\left(\lambda^2-\lambda-1\right)\left(\lambda+\frac{m-1}{m+2}\right)\left(\lambda^2+\frac{m-4}{m+2}\lambda-\frac{m-2}{m+2}\right).$$

It is easy to see that the smallest and the largest zeros of this polynomial are

$$\frac{-m+4-\sqrt{m(5m-8)}}{2(m+2)} \quad \text{and} \quad \frac{1+\sqrt{5}}{2}.$$

In order to determine the extremal polynomial associated with $b_{n-2,max}$ we have to find an eigenvector $\mathbf{d}^{(o)}_{n-2,max}$ of D^{o}_{n-2}, associated with the eigenvalue $\lambda=(1+\sqrt{5})/2$, such that

$$d_0^2+\cdots+d_{2m}^2=1. \tag{5.2}$$

Solve the homogeneous linear system $(D^{(o)}_{n-2}-((1+\sqrt{5})/2)I)\mathbf{d}^{(o)}_{n-2,max}=0$. Set

$$\kappa:=\sqrt{\frac{7+3\sqrt{5}}{5+2\sqrt{5}}}.$$

For the coordinates of $\mathbf{d}^{(o)}_{n-2,max}$ we obtain $d_{2k+1}=0$ for $k=0,\dots,m-1$, and

$$\begin{aligned}
d_0 &= \frac{\kappa}{\sqrt{2}\lambda(1+\sqrt{5})},\\
d_{2k} &= \frac{1+\sqrt{5}}{4}A_k\kappa \quad \text{for} \quad k=1,\dots,m-2,\\
d_{2m-2} &= \frac{2m+3+\sqrt{5}}{2\sqrt{m(m+1)}(1+\sqrt{5})}\kappa,\\
d_{2m} &= \frac{1}{2}\sqrt{\frac{m+2}{m+1}}\kappa.
\end{aligned}$$

A substitution of these values of d_j in (5.2) yields $d_{2m}=\pm(1/2)((m+2)/(m+1))^{1/2}\kappa$. Thus the desired eigenvectors are $\pm\mathbf{d}^{(o)}_{n-2,max}$, where

$$\mathbf{d}^{(o)}_{n-2,max}=\frac{\kappa}{2}\left(\frac{\sqrt{2}}{\lambda(1+\sqrt{5})},0,\dots,0,\frac{2m+3+\sqrt{5}}{(1+\sqrt{5})\sqrt{m(m+1)}},0,\sqrt{\frac{m+2}{m+1}}\right)^T.$$

We shall work with $\mathbf{d}^{(o)}_{n-2,max}$. Relations (3.5) and (3.6) yield $c_0=c_{2m}=\kappa/\sqrt{2}$ and $c_{2k}=\sqrt{2}(1+\sqrt{5})\kappa/4$ for $k=1,\cdots,m-1$. Then the relations (3.1) imply

$$\begin{aligned}
a_0 &= \kappa^2(1+(\frac{3+\sqrt{5}}{4})(m-1),\\
a_{2k} &= \kappa^2((\frac{1+\sqrt{5}}{2})+(\frac{3+\sqrt{5}}{4})(m-k-1)) \quad \text{for} \quad k=1,\dots,m-1,\\
a_{2m} &= \kappa^2/2 \quad \text{and}\\
a_{2k+1} &= 0 \quad \text{for} \quad k=0,\cdots,m-1.
\end{aligned}$$

We obtain the nonnegative cosine polynomial (2.23). On using (3.2) we obtain the nonnegative sine polynomial (2.9).

In order to determine the extremal sine polynomial associated with $b^{(o)}_{n-2,min}$ we find the eigenvector $\mathbf{d}^{(o)}_{n-2,min}$ of $D^{(o)}_{n-2}$, associated with the smallest eigenvalue $\lambda = (-m+4-\sqrt{m(5m-8)})/(2m+4)$.

Set

$$\sigma := \beta\sqrt{1+\frac{m-4}{\sqrt{m(5m-8)}}}.$$

For the coordinates of $\mathbf{d}^{(o)}_{n-2,min}$ we obtain $d_{2k+1}=0$ for $k=0,\dots,m-1$, and

$$\begin{aligned}
d_0 &= -\frac{3m-\sqrt{m(5m-8)}}{4m\sqrt{2}}\alpha\sigma,\\
d_{2k} &= \frac{m+2}{m-\sqrt{m(5m-8)}}\alpha\sigma A_k \quad \text{for} \quad k=1,\dots,m-2,\\
d_{2m-2} &= \frac{m(1-m)+(1+m)\sqrt{m(5m-8)}}{4m\sqrt{m(m+2)}}\sigma,\\
d_{2m} &= \frac{\sigma}{2}.
\end{aligned}$$

By (3.5) and (3.6) we obtain $c_{2k+1}=0$ for $k=0,\cdots,m-1$, and

$$\begin{aligned}
c_0 &= -\frac{\alpha\sigma}{\sqrt{2}},\\
c_{2k} &= \frac{\sqrt{2}(m-2k)(m+\sqrt{m(5m-8)})}{4m(2-m)}\alpha\sigma \quad \text{for} \quad k=1,\dots,m-1,\\
c_{2m} &= -c_0.
\end{aligned}$$

Then expressions (3.1) imply

$$\begin{aligned}
a_0 &= \frac{m(2m-5+\sqrt{m(5m-8)})}{3\sqrt{m(5m-8)}},\\
a_{2k} &= \frac{10k+2k^3-4m-3km^2+m^3}{3(m^2-4)}\\
&\quad +\frac{4k^3(m-1)+m(m^2-4)(2m+1)+k(4+20m-6m^3)}{3(m^2-4)\sqrt{m(5m-8)}},\\
a_{2m} &= -\frac{\alpha^2\sigma^2}{2}.
\end{aligned}$$

We obtain the nonnegative cosine polynomial (2.24) and, by (3.2), the nonnegative sine polynomial (2.10).

5.2. **The extrema of b_{n-2} for $n=2m+2$.**

Proof of Lemma 1. Simple analysis shows that $r_m(y)$ has two positive points of extrema and the values of $r_m(y)$ at these points are negative and positive, respectively. Since $r_m(0)>0$ and $r_m(y)$ is negative for sufficiently large y, then $r_m(y)$ has three positive zeros. On the other hand, $y(\infty)=(3+\sqrt{5})/2$ is strictly greater than the point of local maximum of r_m and $r_m(y(\infty))<0$. This implies the inequality $y(m)<y(\infty)$. It is clear that $r_m(y)$ converges uniformly on the compacts of the complex plane to the polynomial $-y^3+4y^2-4y+1$. Then by a theorem of Hurwitz (see Theorem 3.45 on p. 119 in [15]) the sequence of the largest zeros $y(m)$ of $r_m(y)$ converges to the largest zero $(3+\sqrt{5})/2$ of $-y^3+4y^2-4y+1$ as m diverges.

Proof of Theorem 4. By (3.3), (3.5) and (3.7) we obtain

$$\begin{aligned} b_{n-2} &= \sum_{k=0}^{m}\left(\frac{d_{2k}d_{2m-1}}{2^k q_1 q_2\cdots q_k\sqrt{q_{k+1}}\sqrt{q_m}} + \frac{d_{2k}d_{2m+1}}{2^{k+1}q_1q_2\cdots q_k q_m\sqrt{q_{k+1}}\sqrt{q_{m+1}}}\right)\\ &\quad + \sum_{k=0}^{m}\left(\frac{(-1)^{2^k}d_{2k}d_{2m+1}}{2^k q_1q_2\cdots q_k\sqrt{q_{k+1}}\sqrt{q_{m+1}}} + \frac{d_{2k+1}d_{2m}}{2^k q_1q_2\cdots q_k\sqrt{q_{k+1}}\sqrt{q_{m+1}}}\right)\\ &= 2\sum_{k=0}^{m}\frac{\beta d_{2k}d_{2m-1} + (\gamma+(-1)^{2^k}\alpha)d_{2k}d_{2m+1} + \alpha d_{2k+1}d_{2m}}{\sqrt{(k+1)(k+2)}}, \end{aligned}$$

where α, β and γ are defined by (4.1), (4.5), and (5.1). Thus, the symmetric matrix $D_{n-2}^{(e)}$ associated with the quadratic form which represents b_{n-2} is

$$\begin{pmatrix} 0 & e_1 & 0 & (\alpha+\gamma)A_{m-1} & \cdots & (\gamma-\alpha)A_0\\ e_1 & 0 & e_2 & 0 & \cdots & 0\\ 0 & e_2 & 0 & \beta A_{m-1} & \cdots & \beta A_0\\ \vdots & \vdots & \vdots & & & \\ (\alpha+\gamma)A_1 & 0 & \beta A_1 & & & \\ 0 & \alpha A_0 & 0 & & & \\ (\gamma-\alpha)A_0 & 0 & \beta A_0 & & & \end{pmatrix},$$

where $e_1 = (2\alpha+\gamma)A_m$ and $e_2 = \alpha A_{m-1} + \beta A_m$. Calculations show that the characteristic polynomial of $D_{n-2}^{(e)}$ is

$$det(D_{n-2}^{(e)} - \lambda I) = -\lambda^{2m-4} r_m(\lambda^2),$$

where the polynomial $r_m(y)$ is defined in Lemma 1. The statement of Lemma 1 completes the proof of Theorem 4.

6. Proof of the convergence

In order to establish Theorems 5, 6, and 7 we have to prove that each $K_{n,j}(\theta),\ j = 1, 2, 3,$ is a positive summability kernel. It follows from the way $K_{n,j}(\theta)$ were constructed that they are nonnegative. Obviously, they are also even and properly normalized. Thus all we need to prove is that, for $j = 1, 2, 3$, $\{K_{n,j}(\theta)\}$ converge locally uniformly to zero in $(0, 2\pi)$.

Theorem 10. *The sequences* $\{K_{n,j}(\theta)\}_{n=1}^{\infty},\ j = 1, 2, 3,$ *defined as in the paragraph before the statement of Theorem* 5 *are sequences of positive summability kernels.*

Proof. By differentiation of the well-known formulas

$$\sum_{k=1}^{n}\cos k\theta = \frac{1}{2}\left(-1 + \frac{\sin((2n+1)/2)\theta}{\sin(\theta/2)}\right)$$

and

$$\sum_{k=1}^{n}\sin k\theta = \frac{\sin(n/2)\theta\sin((n+1)/2)\theta}{\sin(\theta/2)},$$

we obtain closed-form expressions for the sums $\sum_{k=1}^{n} k^{\nu}\cos k\theta,\ \nu = 1, 2, 3$. Then expanding the cosine polynomials in terms of linear combinations of the latter sums we obtain the closed-form representations of the kernels $K_{n,j}(\theta)$:

$$K_{n,1}(\theta) = \frac{3((n+2)\sin(n/2)\theta - n\sin((2\theta+n\theta)/2)^2}{4n(n+1)(n+2)\sin^4(\theta/2)},$$

$$\begin{aligned} K_{n,2}(\theta) &= \frac{3}{4(n+1)(n+2)(2n+3)\sin^4(\theta/2)} \\ &\quad \times \{n^2+3n+3-(n^2+3n+2)\cos\theta-(n+2)\cos(n+1)\theta \\ &\qquad +\cos(n+2)\theta+n\cos(n+2)\theta)\} \end{aligned}$$

and

$$K_{n,3}(\theta) = \frac{2-\cos n\theta-\cos(n+1)\theta}{(4n+2)\sin^2(\theta/2)}.$$

Then we obtain immediately the estimates

$$\begin{aligned} \sin^4(\theta/2)K_{n,1}(\theta) &\leq \frac{3(n+1)}{n(n+2)}, \\ \sin^4(\theta/2)K_{n,2}(\theta) &\leq \frac{3(n+2)}{2(n+1)(2n+3)}, \end{aligned}$$

and

$$\sin^2(\theta/2)K_{n,3}(\theta) \leq \frac{1}{n+2}.$$

The fact that, for every $j = 1,2,3$ and every positive integer n, $K_{n,j}(\theta)$, $j = 1,2,3$, are even functions yields the local uniform convergence of the sequences $\{K_{n,j}(\theta)\}_{n=1}^{\infty}$ in $(0,2\pi)$. □

7. Some graphs

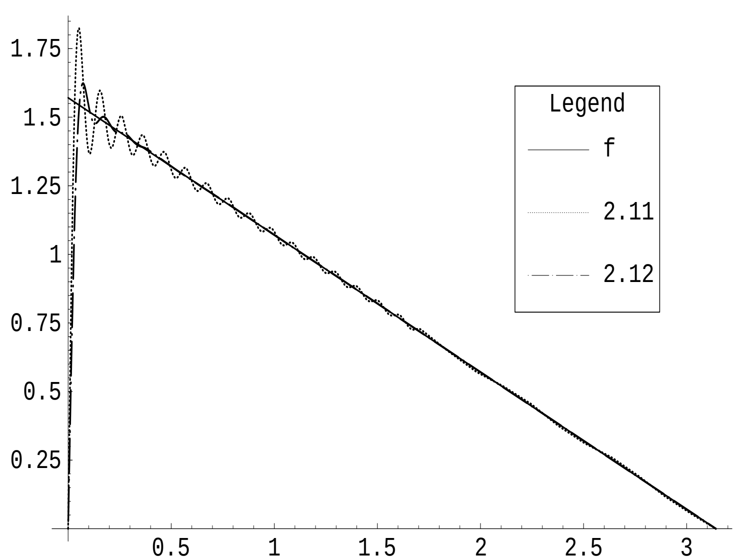

FIGURE 1. The graphs of the function $f(\theta) = (\pi-\theta)/2$ and of the sine polynomials (2.11) and (2.12) for $n = 60$.

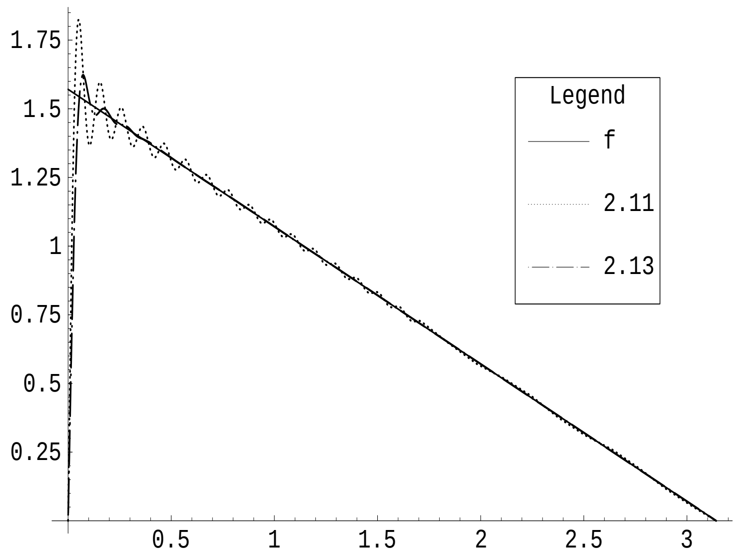

FIGURE 2. The graphs of the function $f(\theta) = (\pi - \theta)/2$ and of the sine polynomials (2.11) and (2.13) for $n = 60$.

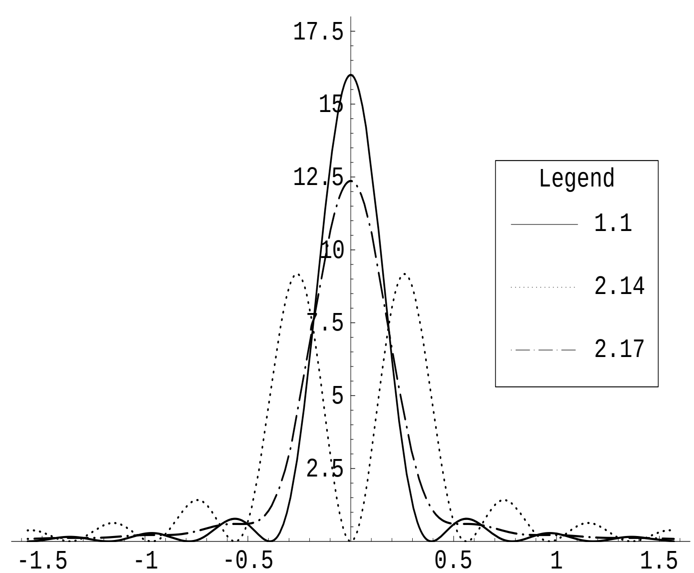

FIGURE 3. The graphs of Fejér's kernel (1.1) and of the cosine polynomials (2.14) and (2.17) for $n = 15$.

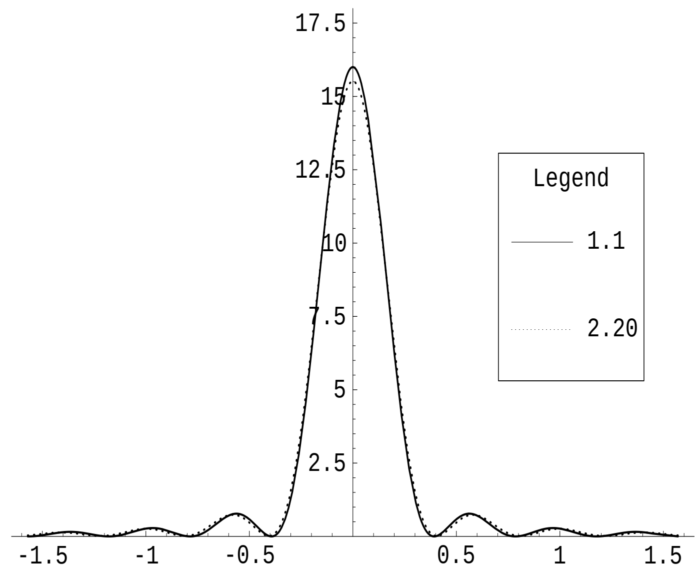

FIGURE 4. The graphs of Fejér's kernel (1.1) and of the cosine polynomial (2.20) for $n = 15$.

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