Monotonicity of zeros of Jacobi polynomials

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Abstract

Denote by $x_{n,k}(\alpha,\beta)$, $k=1,\ldots,n$, the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. It is well known that $x_{n,k}(\alpha,\beta)$ are increasing functions of β and decreasing functions of α. In this paper we investigate the question of how fast the functions $1-x_{n,k}(\alpha,\beta)$ decrease as β increases. We prove that the products $t_{n,k}(\alpha,\beta) := f_n(\alpha,\beta)\,(1-x_{n,k}(\alpha,\beta))$, where $f_n(\alpha,\beta)=2n^2+2n(\alpha+\beta+1)+(\alpha+1)(\beta+1)$, are already increasing functions of β and that, for any fixed $\alpha>-1$, $f_n(\alpha,\beta)$ is the asymptotically extremal, with respect to n, function of β that forces the products $t_{n,k}(\alpha,\beta)$ to increase.

Key words: Zeros, Jacobi polynomials, monotonicity.

1 Introduction and statement of results

The behaviour of the zeros $x_{n,k}(\alpha,\beta)$, $k=1,\ldots,n$, of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, arranged in decreasing order, as functions of the parameters α and β, $\alpha,\beta>-1$, has been of interest for more than a century, since the pioneering contributions of Markov [14] and Stieltjes [18], both published in 1886. Some of the reasons for this interest are the important role that $x_{n,k}(\alpha,\beta)$ play as nodes of Gaussian quadrature formulae and their nice electrostatic interpretation. Stieltjes proved in [18] that, given two fixed charges at the points -1 and 1, with forces $(\beta+1)/2$ and $(\alpha+1)/2$, respectively, and n free unit charges in $(-1,1)$, the energy of the electrostatic field generated by them attains a local minimum when the free charges are located at $x_{n,k}(\alpha,\beta)$. Here the field obeys

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the law of the logarithmic potential which means that all the charges, both the fixed and the free ones, are distributed along infinite wires orthogonal to $(-1,1)$. Szegő [19, Section 6.83] proved that the energy has a unique global minimum which shows that the zeros of the Jacobi polynomial of degree n are the points of stable equilibrium of the energy. Andrey Markov [14], (see also [19, Theorem 6.12.1]) proved that all the zeros $x_{n,k}(\alpha,\beta)$ are increasing functions of β and decreasing functions of α. This fact is intuitively clear from the above electrostatic interpretation since all the charges are positive and repel each other. In this paper we discuss the deeper question of how fast the zeros increase/decrease when the parameters β and α increase from -1 to infinity.

The present piece of research is inspired by the complete solution of corresponding problem about the speed of decrease of the positive zeros $x_{n,k}(\lambda)$ of the ultraspherical polynomial $C_n^\lambda(x)$, as functions of λ, when $\lambda > -1/2$. The solution came after series of papers, published within the last twenty five years, where various conjectures and contributions were made. Instead of asking the straightforward question about the way $x_{n,k}(\lambda)$ decrease, the following equivalent problem was investigated: If the zeros of $C_n^\lambda(x)$ are arranged in decreasing order, which is the extremal function $f_n(\lambda)$ that forces the functions $f_n(\lambda)x_{n,k}(\lambda)$, $k = 1,\dots,[n/2]$, to increase? The exact meaning of the notion "extremal" was described in the introduction of [2] and we recall that reasoning below to justify our choice of the multiplier $f_n(\alpha,\beta)$. The first to pose such a question for the positive zeros of $C_n^\lambda(x)$ was Laforgia who conjectured in [12] that $\lambda x_{n,k}(\lambda)$ increase for $\lambda > 0$. Laforgia had established this result for $\lambda \in (0,1)$ in [11]. Later on, Ismail and Letessier [10] refined the conjecture, restating it with a function that possesses the precise asymptotic behaviour, namely for $f(\lambda) = \sqrt{\lambda}$. Finally, Askey guessed the extremal universal function, that is, the one that does not depend on n, with the above properties. The function turned out to be simply $f(\lambda) = \sqrt{\lambda+1}$ (see [9]). Various contributions to the problem were made by Spigler [17], Ahmed, Muldoon and Spigler [1], Ifantis and Siafarikas [8], Dimitrov [2], while, in 1999, Elbert and Siafarikas [5] proved that $[\lambda+(2n^2+1)/(4n+2)]^{1/2}x_{n,k}(\lambda)$, $k = 1,\dots,[n/2]$, are increasing functions of λ for $\lambda > -1/2$, thus extending the result of Ahmed, Muldoon and Spigler [1] and proving the conjecture of Ismail, Letessier and Askey [10,9]. Finally, it was proved in [3] that the above function $[\lambda+(2n^2+1)/(4n+2)]^{1/2}$ is asymptotically extremal in the sense that there is no function that increases slower than it and forces the products $f_n(\lambda)x_{n,k}(\lambda)$ to increase. Similar questions concerning zeros of Laguerre polynomials were raised and discussed by Natalini and Palumbo [15].

In this paper we state and solve the corresponding question about the zeros of Jacobi polynomials. Surprisingly enough, no attempt has been done to tackle this problem. One of the possible reasons for the lack of results in this direction is that $x_{nk}(\alpha,\beta)$ change sign. This indicates that, instead of considering the zeros of $P_n^{(\alpha,\beta)}(x)$ themselves, it is more reasonable to investigate either $1 - x_{nk}$ or $1 + x_{nk}$. Moreover, a careful inspection of the evolution of the conjectures and results concerning the positive zeros of ultraspherical polynomials, described above, leads to the conclusion that, in order to obtain a sharp result, quantities that obey nice asymptotic behaviour, as the parameter diverges, must be considered. Such an asymptotic formula for the zeros of Jabobi polynomials is (see [19, formula (6.71.11)])

$$\lim_{\beta\to\infty} \beta(1 - x_{nk}(\alpha,\beta)) = 2\ x_{n,n-k+1}(\alpha), \tag{1.1}$$

where $x_{nj}(\alpha)$ are the zeros of the Laguerre polynomial $L_n^{(\alpha)}(x)$, arranged in decreasing order. These observations, together with the fact that $1-x_{nk}(\alpha,\beta)$ decrease when β increases, already suggest that we search for a function $f_n(\alpha,\beta)$ that forces the products $f_n(\alpha,\beta)(1 - x_{nk}(\alpha,\beta))$ to increase. It is natural to require that, for any fixed $\alpha \in (-1,\infty)$, f_n is positive and smooth function of β, for $\beta \in (-1,\infty)$.

Consider the quantities $Z_{nk}(\alpha,\beta) = f_{nk}(\alpha,\beta)(1-x_{nk}(\alpha,\beta))$ as functions of β, where $f_{nk}(\alpha,\beta)$ are also positive and differentiable functions of β, for any fined n, k and $\alpha \in (-1,\infty)$. What additional necessary conditions for $f_{nk}(\alpha,\beta)$ the inequalities $\partial Z_{nk}(\alpha,\beta)/\partial\beta \geq 0$ imply? Since these are equivalent to

$$0 \leq \frac{\partial Z_{nk}(\alpha,\beta)}{\partial\beta} = \frac{\partial f_{nk}(\alpha,\beta)}{\partial\beta}(1 - x_{nk}(\alpha,\beta)) - f_{nk}(\alpha,\beta)\frac{\partial x_{nk}(\alpha,\beta)}{\partial\beta},$$

$f_{nk}(\alpha,\beta) > 0$, $(1 - x_{nk}(\alpha,\beta)) > 0$, and $\partial x_{nk}(\alpha,\beta)/\partial\beta$, then we must have

$$\frac{\partial \ln f_{nk}(\alpha,\beta)}{\partial\beta} > -\frac{\partial \ln(1 - x_{nk}(\alpha,\beta))}{\partial\beta}. \tag{1.2}$$

Thus, if we search for positive functions $f_{nk}(\alpha,\beta)$, $n \in \mathbb{N}$, $k = 1,\ldots,n$, that are smooth with respect to β and force the corresponding products $Z_{nk}(\alpha,\beta)$ to increase with β, then the best possible are those which satisfy (1.2). Observe that this problem is intractable because it is equivalent to determine explicitly the logarithmic derivatives of all $(1 - x_{nk}(\alpha,\beta))$ and this latter task itself requires to find explicitly all zeros of all Jacobi polynomials. Therefore, we reduce the problem to find, for any fixed $n \in \mathbb{N}$ and $\alpha \in (-1,\infty)$, a positive function $f_n(\alpha,\beta)$ which forces the products $t_{n,k}(\alpha,\beta) = f_n(\alpha,\beta)\,(1 - x_{n,k}(\alpha,\beta))$, $k = 1,\ldots,n$, to increase as β increases from -1 to infinity. Observe that if we were able to determine the functions $f_{nk}(\alpha,\beta)$, then $f_n(\alpha,\beta)$ would have been a piecewise smooth function given by

$$\frac{\partial \ln f_n(\alpha,\beta)}{\partial\beta} = max_{1\leq k\leq n}\frac{\partial \ln f_{nk}(\alpha,\beta)}{\partial\beta}.$$

Hence, if we require that $f_n(\alpha,\beta)$ is a smooth function of β with the property that $t_{nk}(\alpha,\beta)$ increase with β, then the best possible choice of f_n is the one for which its logarithmic derivative with respect to β is the smallest possible.

Our main result read as follows.

Theorem 1 *For every $n \in \mathbb{N}$ and $k,\ k = 1,\dots,n$, the products*

$$f_n(\alpha,\beta)\ (1 - x_{n,k}(\alpha,\beta)),$$

where $f_n(\alpha,\beta) := 2n^2 + 2n(\alpha+\beta+1) + (\alpha+1)(\beta+1)$, are increasing functions of β, for $\beta \in (-1,\infty)$.

A simple argument for symmetry (see formula (2.4) below) immediately implies:

Corollary 1 *For every $n \in \mathbb{N}$ and $k,\ k = 1,\dots,n$, the products*

$$\Big(2n^2 + 2n(\alpha+\beta+1) + (\alpha+1)(\beta+1)\Big)\ (1 + x_{n,k}(\alpha,\beta))$$

are increasing functions of α, for $\alpha \in (-1,\infty)$.

In order to justify the sharpness of Theorem 1, observe that it can be reformulated, stating that the products

$$g_n(\alpha,\beta)\,(1 - x_{n,k}(\alpha,\beta)),$$

where

$$g_n(\alpha,\beta) = \beta + n + \frac{\alpha+1}{2} + \frac{1-\alpha^2}{2(2n+\alpha+1)},$$

are increasing functions of β in $(-1,\infty)$.

We employ the method developed in [3] and based on the classical Routh-Hurwitz stability criterion to prove that the function $g_n(\alpha,\beta)$ is asymptotically extremal with respect to n. The result is the following:

Theorem 2 *Let $n \in \mathbb{N}$, $\alpha > -1$ and $h_n(\alpha,\beta)$, considered as a function of β, be positive and continuously differentiable for $\beta \in (-1,\infty)$. If the products $h_n(\alpha,\beta)(1 - x_{nk}(\alpha,\beta))$, $k = 1,\dots,n$, are increasing functions of β in $(-1,\infty)$, then*

$$\frac{\partial}{\partial\beta}\ln h_n(\alpha,\beta) > \frac{1}{n+\alpha+\beta+1}. \tag{1.3}$$

Moreover, if $h_n(\alpha,\beta) = \beta + n + (\alpha+1)/2 + d_n(\alpha)$, then

$$d_n(\alpha) < \frac{(\alpha+1)(\alpha+2)}{2(n+\alpha+1)}. \tag{1.4}$$

The fact that the extremal function with the desired property must posses the smallest possible logarithmic derivative and inequality (1.3) imply that the extremal h_n must be a linear function of β. This justifies the choice of $h_n(\alpha,\beta)$, as given in the second statement of Theorem 2. A comparison of the explicit form of the function $g_n(\alpha,\beta)$ and inequality (1.4) shows that $g_n(\alpha,\beta)$ is asymptotically extremal. Indeed, obviously $g_n(\alpha,\beta) - h_n(\alpha,\beta) < 0$ for all admissible n, α and β and this difference behaves as $\mathcal{O}(1/n)$, as n goes to infinity provided α is fixed. Finally we mention an immediate but interesting consequence of Theorem 1 and the asymptotic formula (1.1).

Corollary 2 *Let $n \in \mathbb{N}$, $\alpha, \beta > -1$. Then the inequalities*

$$\{2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)\}(1-x_{nk}(\alpha,\beta)) < 2(2n+\alpha+1)x_{n,n-k+1}(\alpha)$$

hold for $k = 1, \ldots, n$.

2 Preliminary technical results

Recall that the Jacobi polynomials can be represented by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} F(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}) \tag{2.1}$$

in terms of the hypergeometric function

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(\gamma)_k$ denotes the Pochhammer symbol, defined by $(\gamma)_k = \gamma \cdots (\gamma+k-1)$, for $k \in \mathbb{N}$, and $(\gamma)_0 := 1$.

Since $y(z) = F(a,b;c;z)$ satisfies the differential equation

$$z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0,$$

the Jacobi polynomial $Y(x) = P_n^{(\alpha,\beta)}(x)$ is a solution of

$$(1-x^2)Y'' + [\beta-\alpha-(\alpha+\beta+2)x]Y' + n(n+\alpha+\beta+1)Y = 0.$$

Introduce the polynomial

$$q_n^{(\alpha,\beta)}(y) = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(2y+1).$$

Then

$$q_n^{(\alpha,\beta)}(y) = \sum_{k=0}^{n} \binom{n}{k} \frac{(n+\alpha+\beta+1)_k}{(\alpha+1)_k} y^k \tag{2.2}$$

and its zeros $y_{n,k}(\alpha,\beta)$ are given by $y_{n,k}(\alpha,\beta) = (x_{n,k}(\alpha,\beta)-1)/2$. Moreover, the polynomials $q_n^{(\alpha,\beta)}(y)$ are orthogonal in $(-1,0)$ and all $y_{n,k}(\alpha,\beta)$ belong to this interval.

We shall need some additional information about functions whose zeros coincide with

$$\widetilde{t}_{nk}(\alpha,\beta) = t_{nk}(\alpha,\beta)/2 = f_n(\alpha,\beta)(1-x_{nk}(\alpha,\beta))/2.$$

Recall first (see [19, p. 67]) that the function $u(x) = (1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}} P_n^{(\alpha,\beta)}(x)$ satisfies the Sturm-Liouville differential equation

$$\frac{d^2u(x)}{dx^2} + \lambda(x;\alpha,\beta)u(x) = 0,$$

with

$$\lambda(x;\alpha,\beta) = \frac{1-\alpha^2}{4(1-x)^2} + \frac{1-\beta^2}{4(1+x)^2} + \frac{n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)/2}{1-x^2}.$$

Then a straightforward change of variables implies that

$$u(z) = z^{\frac{\alpha+1}{2}}(1-z)^{\frac{\beta+1}{2}} P_n^{(\alpha,\beta)}(1-2z),$$

whose zeros in $(0,1)$ are $z_{nk} = (1-x_{nk})/2$, is a solution of

$$\frac{d^2u(z)}{dz^2} + \Lambda(z;\alpha,\beta)u(z) = 0,$$

with

$$\begin{aligned}\Lambda(z;\alpha,\beta) = &\frac{\alpha+\beta+2}{2z(1-z)} + \frac{n(n+\alpha+\beta+1)}{z(1-z)} + \frac{\alpha+1-z(\alpha+\beta+2)}{2z^2(1-z)} \\ &- \frac{\alpha+1-z(\alpha+\beta+2)}{2z(1-z)^2} - \frac{(\alpha+1-z(\alpha+\beta+2))^2}{4z^2(1-z)^2}.\end{aligned}$$

Thus we immediately conclude that:

Lemma 1 *The function $U(t) = u(t/f)$,*

$$U(t) = \Big(\frac{t}{f}\Big)^{\frac{\alpha+1}{2}} \Big(1 - \frac{t}{f}\Big)^{\frac{\beta+1}{2}} P_n^{(\alpha,\beta)}\Big(1 - 2\frac{t}{f}\Big),$$

where $f = f_n(\alpha, \beta)$, is a solution of the differential equation

$$\frac{d^2U(t)}{dt^2} + \widetilde{\Lambda}(t; \alpha, \beta)\; U(t) = 0, \tag{2.3}$$

where

$$\widetilde{\Lambda}(t; \alpha, \beta) = \frac{1}{[f_n(\alpha, \beta)]^2}\; \Lambda\left(\frac{t}{f_n(\alpha, \beta)}; \alpha, \beta\right)$$

that is,

$$\begin{aligned}\widetilde{\Lambda}(t; \alpha, \beta) = &-\frac{\alpha+\beta+2}{2t(t-f)} - \frac{n(n+\alpha+\beta+1)}{t(t-f)} + \frac{t(\alpha+\beta+2)-(\alpha+1)f}{2t(t-f)^2}\\ &+\frac{t(\alpha+\beta+2)-(\alpha+1)f}{2t^2(t-f)} - \frac{(t(\alpha+\beta+2)-(\alpha+1)f)^2}{4t^2(t-f)^2}.\end{aligned}$$

Moreover, the zeros of $U(t)$ are 0, $f_n(\alpha, \beta)$ and $\widetilde{t}_{nk}(\alpha, \beta)$, $k = 1, \ldots, n$.

We shall need the explicit expressions for integrals of the form $I_\nu = I_\nu(n, \alpha, \beta)$, where

$$I_\nu = \int_{-1}^{1} (1-x)^\alpha (1+x)^{\beta-\nu} \{P_n^{(\alpha,\beta)}(x)\}^2 dx,\ \beta > \nu - 1.$$

More precisely, we shall make use of I_0, I_1 and I_2. In fact, I_0 and I_1 are known. The integral

$$I_0 = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)},$$

is given in [19, p. 68] and

$$I_1 = \frac{2^{\alpha+\beta}}{n!\beta} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}$$

is a consequence of formula 7.391(5) in [7] and the relation (see [19, p. 59])

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x). \tag{2.4}$$

Observe that the explicit integral expression

$$\int_{-1}^{1} (1-x)^\rho (1+x)^\beta P_n^{(\alpha,\beta)}(x)\; dx = \frac{2^{\beta+\rho+1}\Gamma(\rho+1)\Gamma(n+\beta+1)\Gamma(n+\alpha-\rho)}{n!\Gamma(\alpha-\rho)\Gamma(n+\beta+\rho+2)}$$

that appears as item 7.391(4) in [7], and (2.4) yield

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\rho} P_n^{(\alpha,\beta)}(x)\ dx$$
$$= (-1)^n \frac{2^{\alpha+\rho+1}\Gamma(\rho+1)\Gamma(n+\alpha+1)\Gamma(n+\beta-\rho)}{n!\Gamma(\beta-\rho)\Gamma(n+\alpha+\rho+2)}, \tag{2.5}$$

and the latter holds for any $n \in \mathbb{N}$, $\alpha, \rho > -1$.

Lemma 2 *For every $n \in \mathbb{N}$ and $\alpha > -1, \beta > 1$ we have*

$$I_2 = \frac{2^{\alpha+\beta-1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \frac{2n(n+\alpha+\beta+1)+(\alpha+\beta)(\beta+1)}{\beta(\beta-1)(\beta+1)}.$$

Proof The relation (2.4) and the explicit expression (2.1) for $P_n^{(\alpha,\beta)}(x)$, yield

$$P_n^{(\alpha,\beta)}(x) = (-1)^n \frac{(\beta+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (n+\alpha+\beta+1)_k}{(\beta+1)_k} \frac{1}{k!} \frac{(1+x)^k}{2^k}.$$

Hence,

$$\begin{aligned} I_\nu &= \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta-\nu}\{P_n^{(\alpha,\beta)}(x)\}^2 dx \\ &= \frac{(-1)^n(\beta+1)_n}{n!} \\ &\quad \times \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{2^k k!(\beta+1)_k} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta+k-\nu} P_n^{(\alpha,\beta)}(x) dx. \end{aligned}$$

The explicit forms of I_1 and I_2 are immediate consequences of this expression and of (2.5). In particular, for I_2 we have

$$\begin{aligned} I_2 &= (-1)^n \frac{(\beta+1)_n}{n!} \Big\{ \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta-2} P_n^{(\alpha,\beta)}(x) dx \\ &\qquad - \frac{n(n+\alpha+\beta+1)}{2(\beta+1)} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta-1} P_n^{(\alpha,\beta)}(x) dx \Big\}. \\ &= (-1)^n \frac{(\beta+1)_n}{n!} \Big\{ (-1)^n \frac{2^{\alpha+\beta-1}}{n!} \frac{\Gamma(\beta-1)\Gamma(n+\alpha+1)\Gamma(n+2)}{\Gamma(2)\Gamma(n+\alpha+\beta)} \\ &\qquad - \frac{n(n+\alpha+\beta+1)}{2(\beta+1)} (-1)^n \frac{2^{\alpha+\beta}}{n!} \frac{\Gamma(\beta)\Gamma(n+\alpha+1)\Gamma(n+1)}{\Gamma(1)\Gamma(n+\alpha+\beta+1)} \Big\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{\alpha+\beta-1}(\beta+1)_n}{n!}\frac{\Gamma(\beta-1)\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)} \\
&\qquad\qquad \times \left\{(n+1)(n+\alpha+\beta) - \frac{n(n+\alpha+\beta+1)}{\beta+1}(\beta-1)\right\} \\
&= 2^{\alpha+\beta-1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}\frac{2n(n+\alpha+\beta+1)+(\alpha+\beta)(\beta+1)}{\beta(\beta+1)(\beta-1)}.
\end{aligned}
$$

3 Proof of Theorem 1 for $\beta \in (-1, 1)$

The proof of our main result for $\beta \in (-1, 1)$ is rather straightforward. It follows immediately from the following two statements.

Theorem 3 *Let, for some fixed values of $n \in \mathbb{N}$ and $\alpha > -1$, the function $f_n(\alpha, \beta)$ be defined and positive for $\beta \in I$, where I is an interval, $I \subset (-1, \infty)$. Suppose further that $f_\beta = \partial f_n(\alpha, \beta)/\partial\beta$ exists, it is continuous and positive for $\beta \in I$. Then*

$$
\widetilde{t}_{nk}(\alpha, \beta) = f_n(\alpha, \beta)\frac{1 - x_{n,k}(\alpha, \beta)}{2}
$$

are increasing function of β in I, provided

$$
\begin{aligned}
&(2n+\alpha+\beta+1)\ t^2 \qquad (3.1)\\
&+[(2n(n+\alpha+\beta+1)+(\alpha+\beta)(\beta+1))f_\beta - (4n+2\alpha+\beta+2)f]\ t \\
&+[(2n+\alpha+1)f - (2n^2+2n(\alpha+\beta+1)+(\alpha+1)(\beta+1))]\ f\ f_\beta \ < \ 0
\end{aligned}
$$

for every $0 < t < f$.

Proof The statement of the theorem follows from Sturm's comparison theorem on solutions of Sturm-Liouville differential equation. The version we need is Theorem 1.82.1 in [19]. It implies that the zeros $\widetilde{t}_{nk}$ of the solution of the differential equation (2.3) are increasing functions of the parameter β provided the partial derivative of $\widetilde{\Lambda}(t; \alpha, \beta)$ with respect to β is negative for every $t \in (0, f)$ when $\beta \in I$. Thus, the straightforward calculation

$$
\begin{aligned}
\frac{\partial\widetilde{\Lambda}(\alpha,\beta)}{\partial\beta} = &\frac{1}{2t(f-t)^3}\{(2n+\alpha+\beta+1)t^2 \\
&+[(2n(n+\alpha+\beta+1)+(\alpha+\beta)(\beta+1))f_\beta - (4n+2\alpha+\beta+2)f]\ t \\
&+[(2n+\alpha+1)f - (2n^2+2n(\alpha+\beta+1)+(\alpha+1)(\beta+1))]\ f\ f_\beta\}
\end{aligned}
$$

completes the proof.

Lemma 3 *If* $f_n(\alpha,\beta) = 2n^2+2n(\alpha+\beta+1)+(\alpha+1)(\beta+1)$ *and* $\beta \in (-1,1)$, *then inequality* (3.1) *holds.*

Proof Substituting the explicit form of $f_n(\alpha,\beta)$ on the left-hand side of (3.1), we obtain the polynomial

$$r(t) = a_n(\alpha,\beta)\ t^2 - b_n(\alpha,\beta)\ t,$$

where $a_n(\alpha,\beta) = 2n+\alpha+\beta+1$ and

$$b_n(\alpha,\beta) = \{4n^3+6n^2(\alpha+\beta+1)+2n(\alpha^2+3\alpha+3+3(\alpha+1)\beta)+(\alpha+1)(\alpha+2)(\beta+1)\}.$$

Observe that the free coefficient of the quadratic in (3.1) vanishes because of the proper choice of the function f. Obviously the leading coefficient a_n is always positive and the zeros of $r(t)$ are 0 and $b_n(\alpha,\beta)/a_n(\alpha,\beta)$. Moreover, the latter positive zero exceeds $f_n(\alpha,\beta)$ if and only if $a_n f_n < b_n$. On the other hand, this inequality is equivalent to

$$\frac{(2n+\alpha+1)(\beta^2-1)}{2n+\alpha+\beta+1} < 0$$

which itself is nothing but the requirement $-1 < \beta < 1$. Hence, for these values of β, the binomial $r(t)$ is negative for every $t \in (0, f)$.

4 Proof of Theorem 1 for $\beta \in (1,\infty)$

Consider a parametric family of Sturm-Liouville equations of the form

$$y''(x) + F(x,\mu)y(x) = 0,$$

for $0 < x < \delta$, where the parameter μ varies in certain interval and the function $F(x,\mu)$ is continuously differentiable with respect to both variables, so that $\partial F(x;\mu)/\partial\mu$ is integrable function of x in $(0,\delta)$. Then the solution $y = y(x,\mu)$ is also smooth with respect to x and μ. Moreover, if the zeros of the solution y are distinct, then each such a zero c is a smooth function of the parameter μ. Elbert and Muldoon [4] obtained a beautiful formula for the derivative $c'(\mu)$ of any zero c of a solution of y provided the above requirements are fulfilled and, in addition, the solution satisfies either of the requirements $y(0,\mu) = 0$ or $y'(0,\mu) = 0$, where the last derivative is with respect to the first variable. The formula reads as

$$\Big[\frac{dy(x,\mu)}{dx}\Big|_{x=c}\Big]^2\ c'(\mu) = -\int_0^c \frac{\partial F(x;\mu)}{\partial\mu}\ [y(x,\mu)]^2 dx.$$

Having in mind that $\partial\widetilde{\Lambda}(t;\alpha,\beta)/\partial\beta$ is continuous in $(0,f)$, that $U(0)=0$, and applying this formula for the derivatives of the zeros $\widetilde{t}_{nk}$ of the solution $U(t)$ of the differential equation (2.3), we obtain

$$\Big[\frac{\partial U(t;\alpha,\beta)}{\partial t}\Big|_{t=\widetilde{t}_{nk}}\Big]^2\ \frac{\partial \widetilde{t}_{nk}(\alpha,\beta)}{\partial\beta} = -\int\limits_0^{\widetilde{t}_{nk}} \frac{\partial\widetilde{\Lambda}(t;\alpha,\beta)}{\partial\beta}\ U^2(t;\alpha,\beta)\,dt.$$

Substituting the explicit expressions of $U(t;\alpha,\beta)$ and $\widetilde{\Lambda}(t;\alpha,\beta)$ into the right-hand side integral

$$-\int\limits_0^{\widetilde{t}_{nk}} \frac{\partial\widetilde{\Lambda}(t;\alpha,\beta)}{\partial\beta}\ U^2(\alpha,\beta;t)dt,$$

we obtain

$$\Big[\frac{\partial U}{\partial t}\Big]^2\frac{\partial\widetilde{t}_{nk}}{\partial\beta} = -\int\limits_0^{\widetilde{t}_{nk}} \frac{r(t)}{2t(f-t)^3}\Big(\frac{t}{f}\Big)^{\alpha+1}\Big(1-\frac{t}{f}\Big)^{\beta+1}\Big\{P_n^{(\alpha,\beta)}\Big(1-2\frac{t}{f}\Big)\Big\}^2dt,$$

where $r(t)$ is exactly the polynomial defined in the proof of Lemma 3. Let us investigate the function

$$\Phi(\tau) = -\int\limits_0^{\tau} \frac{r(t)}{2t(f-t)^3}\Big(\frac{t}{f}\Big)^{\alpha+1}\Big(1-\frac{t}{f}\Big)^{\beta+1}\Big\{P_n^{(\alpha,\beta)}\Big(1-2\frac{t}{f}\Big)\Big\}^2dt.$$

For, observe that latter integrand changes sign if and only if $r(t)$ does. However, the discussion in the proof of Lemma 3 shows that it happens only when $t=b_n/a_n$. On the other hand, recalling again the investigation of the behaviour of this quotient and the fact that now $\beta>1$, we see that in this case $b_n/a_n<f$. Summarizing, we conclude that $\Phi(\tau)$ is an increasing function of τ in $(0,b_n/a_n)$ and it decreases in $(b_n/a_n,f)$.

We shall prove that $\Phi(f)=0$. Performing the change of variables $t=f(1-x)/2$ in the integral that represents $\Phi(f)$, we obtain

$$\begin{aligned}\Phi(f) = A&\int\limits_{-1}^{1}(1-x)^{\alpha+1}(1+x)^{\beta-2}\{P_n^{(\alpha,\beta)}(x)\}^2dx\\ &+B\int\limits_{-1}^{1}(1-x)^{\alpha+1}(1+x)^{\beta-1}\{P_n^{(\alpha,\beta)}(x)\}^2dx,\end{aligned}$$

where

$$A=\frac{(1-\beta^2)(2n+\alpha+1)}{2^{\alpha+\beta+1}f^2},\quad B=\frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+2}f}.$$

Then the straightforward calculation

$$\int_{-1}^{1}(1-x)^{\alpha+1}(1+x)^{\mu}\{P_n^{(\alpha,\beta)}(x)\}^2dx =$$
$$2\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\mu}\{P_n^{(\alpha,\beta)}(x)\}^2dx - \int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\mu+1}\{P_n^{(\alpha,\beta)}(x)\}^2dx$$

immediately yields

$$\begin{aligned}\Phi(f) = 2A &\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta-2}\{P_n^{(\alpha,\beta)}(x)\}^2dx \\ &+(2B-A)\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta-1}\{P_n^{(\alpha,\beta)}(x)\}^2dx \\ &-B\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\{P_n^{(\alpha,\beta)}(x)\}^2dx\end{aligned}$$

Thus, $\Phi(f) = 2AI_2 + (2B-A)I_1 - BI_0 = 0$.

Therefore, $\Phi(\tau) > 0$ for every $\tau \in (0, f)$, and in particular $\Phi(\widetilde{t}_{nk}) > 0$. This shows that $\widetilde{t}_{nk}(\alpha,\beta)$ are increasing functions of β for $\beta > 1$.

5 Proof of Theorem 2

The fact that $f_n(\alpha,\beta)$ is extremal in the sense described in the introduction is justified by at least three arguments. The first one is that $\Phi(f_n(\alpha,\beta)) = 0$. The second one, that the authors discovered first, was an application, for $\beta \in (-1,1)$, of the technique used by Ahmed, Muldoon and R. Spigler [1] for the extremal function in the ultraspherical case. This involves rather lengthy technical details. Despite that these arguments led us to the correct guess, they are somehow intuitive and not rigorous at all.

Because of that we provide a completely correct argument. We apply the method developed in [3] and use the notations adopted there. The ideas in [3] we based on the stability criterion of Routh-Hurwitz. We refer to Gantmacher [6, Chapter 15] and Marden [13, Chapter 9] for comprehensive information on this classical topic. We shall provide some definitions and formulate the Hurwitz theorem. With every polynomial with real coefficients

$$f(z) = f_n z^n + f_{n-1}z^{n-1} + f_{n-2}z^{n-2} + f_{n-3}z^{n-3} + \cdots, \; f_n \neq 0,$$

we associate a Hurwitz matrix which is formed as follows. Set $f_{-1} = f_{-2} = \cdots = 0$ and construct the two line block

$$\begin{pmatrix} f_{n-1} & f_{n-3} & \cdots \\ f_n & f_{n-2} & \cdots \end{pmatrix},$$

where the first line contains $f_{n-2k-1},\ k = 0, 1, \ldots,$ and the second line is composed by the coefficients $f_{n-2k},\ k = 0, 1, \ldots,$ of $f(z)$. Then the Hurwitz matrix $H(f)$ of $f(z)$ is composed by the above block in its first two lines, the next two lines of $H(f)$ contain the same block shifted one position to the right, the fifth and the sixth lines contain this block shifted two positions to the right, and so forth. Thus

$$H(f) = \begin{pmatrix} f_{n-1} & f_{n-3} & f_{n-5} & \ldots & 0 \\ f_n & f_{n-2} & f_{n-4} & \ldots & 0 \\ 0 & f_{n-1} & f_{n-3} & \ldots & 0 \\ 0 & f_n & f_{n-2} & \ldots & 0 \\ . & . & . & \ldots & . \end{pmatrix}.$$

The polynomial $f_n(z) = f_n z^n + f_{n-1} z^{n-1} + \cdots + f_0$ with real coefficients f_j, and with positive leading coefficient f_n is called Hurwitz or stable if all its zeros have negative real parts. The following is the celebrated Hurwitz theorem which is sometimes called the Routh-Hurwitz criterion.

Theorem A *The polynomial $f_n(z)$ is stable if and only if the first n principal minors of the corresponding Hurwitz matrix $H(f)$ are positive.*

We shall say that the polynomials $h(z)$ and $g(z)$ of degree m form a positive pair if their leading coefficients are positive and their zeros $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ are distinct, real, negative and interlace in the following way:

$$y_m < x_m < y_{m-1} < x_{m-1} < \cdots < y_1 < x_1.$$

We shall succinctly denote the latter by $\bar{y} \prec \bar{x}$.

Theorem B *The polynomial $f(z) = h(z^2) + zg(z^2)$ is a Hurwitz polynomial if and only if $h(z)$ and $g(z)$ form a positive pair.*

This result appears as Theorem 13 on p. 228 in [6]. It is an immediate consequence of Theorem A and of the Theorem of Hermite-Biehler (see Obrechkoff [16]).

Consider the sequence $\{p_n(x;\tau)\}$ of parametric polynomials which are orthogonal on the interval $x \in (c, d)$ when $\tau \in (p, q)$ and whose coefficients are continuous functions of τ. Suppose the leading coefficients of $p_n(x;\tau)$ are positive. We shall denote by $\zeta_k(\tau)$,

$$c < \zeta_n(\tau) < \zeta_{n-1}(\tau) < \cdots < \zeta_1(\tau) < d,$$

the zeros of $p_n(x;\tau)$ arranged in decreasing order. Let

$$p_n(x;\tau) = a_0(\tau) + a_1(\tau)(x-d) + \cdots + a_n(\tau)(x-d)^n, \quad a_n(\tau) > 0, \quad (5.1)$$

be the Taylor expansion of $p_n(x;\tau)$ at d. Since the zeros $\zeta_k(\tau),\ k = 1, \ldots, n$ of $p_n(x;\tau)$ are distinct and belong to (c,d), then all the coefficients $a_j(\tau),\ j = 0, \ldots, n$, are positive. Let $q_n(x;\tau)$ be the polynomial

$$q_n(x;\tau) = a_0(\tau) + a_1(\tau)x + \cdots + a_n(\tau)x^n,$$

and

$$\tilde{q}_n(x;\tau) = a_0(\tau)x^n + \cdots + a_1(\tau)x + a_n(\tau)$$

be its inverse. Denote by $H(p_n;\tau_1,\tau_2)$ the Hurwitz matrix associated with the polynomial

$$f_{2n+1}(x;\tau_1,\tau_2) := q_n(x^2;\tau_1) + xq_n(x^2;\tau_2).$$

We have

$$H(p_n;\tau_1,\tau_2) = \begin{pmatrix} a_n(\tau_1) & a_{n-1}(\tau_1) & a_{n-2}(\tau_1) & \ldots & 0 \\ a_n(\tau_2) & a_{n-1}(\tau_2) & a_{n-2}(\tau_2) & \ldots & 0 \\ 0 & a_n(\tau_1) & a_{n-1}(\tau_1) & \ldots & 0 \\ 0 & a_n(\tau_2) & a_{n-1}(\tau_2) & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \end{pmatrix}.$$

Similarly, $\tilde{H}(p_n;\tau_1,\tau_2)$ denotes the Hurwitz matrix associated with

$$f^*_{2n+1}(x;\tau_1,\tau_2) := \tilde{q}_n(x^2;\tau_1) + x\tilde{q}_n(x^2;\tau_2).$$

Thus

$$\tilde{H}(p_n;\tau_1,\tau_2) = \begin{pmatrix} a_0(\tau_1) & a_1(\tau_1) & a_2(\tau_1) & \ldots & 0 \\ a_0(\tau_2) & a_1(\tau_2) & a_2(\tau_2) & \ldots & 0 \\ 0 & a_0(\tau_1) & a_1(\tau_1) & \ldots & 0 \\ 0 & a_0(\tau_2) & a_1(\tau_2) & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \end{pmatrix}.$$

For any $j,\ 1 \le j \le 2n+1$, denote by $\Delta_j(p_n;\tau_1,\tau_2)$ and $\tilde{\Delta}_j(p_n;\tau_1,\tau_2)$ the j-th principal minor of $H(p_n;\tau_1,\tau_2)$ and $\tilde{H}(p_n;\tau_1,\tau_2)$, respectively. For the first few j we have

$$\Delta_1(p_n;\tau_1,\tau_2)=a_n(\tau_1),\quad \Delta_2(p_n;\tau_1,\tau_2)=\begin{vmatrix} a_n(\tau_1) & a_{n-1}(\tau_1)\\ a_n(\tau_2) & a_{n-1}(\tau_2)\end{vmatrix},$$

$$\Delta_3(p_n;\tau_1,\tau_2)=\begin{vmatrix} a_n(\tau_1) & a_{n-1}(\tau_1) & a_{n-2}(\tau_1)\\ a_n(\tau_2) & a_{n-1}(\tau_2) & a_{n-2}(\tau_2)\\ 0 & a_n(\tau_1) & a_{n-1}(\tau_1)\end{vmatrix},$$

$$\tilde{\Delta}_1(p_n;\tau_1,\tau_2)=a_0(\tau_1),\quad \tilde{\Delta}_2(p_n;\tau_1,\tau_2)=\begin{vmatrix} a_0(\tau_1) & a_1(\tau_1)\\ a_0(\tau_2) & a_1(\tau_2)\end{vmatrix},$$

$$\tilde{\Delta}_3(p_n;\tau_1,\tau_2)=\begin{vmatrix} a_0(\tau_1) & a_1(\tau_1) & a_2(\tau_1)\\ a_0(\tau_2) & a_1(\tau_2) & a_2(\tau_2)\\ 0 & a_0(\tau_1) & a_1(\tau_1)\end{vmatrix}.$$

Thus, we formulate one of the principal results in [3]:

Theorem 4 *Let the coefficients $a_k(\tau)$ in the representation* (5.1) *of the parametric orthogonal polynomial $p_n(x;\tau)$ be continuous functions of τ. Then:*

(i) The inequalities

$$\zeta_k(\tau_2)<\zeta_k(\tau_1),\quad k=1,\dots,n, \tag{5.2}$$

hold for any τ_2 in a sufficiently small neighbourhood of τ_1 if and only if $\Delta_j(p_n,\tau_1,\tau_2)>0$ for $j=1,\dots,2n+1$;

(ii) The inequalities

$$\zeta_k(\tau_1)<\zeta_k(\tau_2),\quad k=1,\dots,n, \tag{5.3}$$

hold for any τ_2 in a sufficiently small neighbourhood of τ_1 if and only if $\tilde{\Delta}_j(p_n,\tau_1,\tau_2)>0$ for $j=1,\dots,2n+1$.

The proof of Theorem 2 follows almost immediately from the latter statement and some easy calculations. It was already mentioned in the introduction that the zeros of the polynomial $q_n^{(\alpha,\beta)}(y)$, defined by (2.2), are $y_{n,k}(\alpha,\beta)=(x_{n,k}(\alpha,\beta)-1)/2$. Thus the zeros of the polynomial $Q_n^{(\alpha,\beta)}(y):=q_n^{(\alpha,\beta)}(y/h)$, with $h=h_n(\alpha,\beta)$, are precisely $h_n(\alpha,\beta)\,(x_{n,k}(\alpha,\beta)-1)/2$, $k=1,\dots,n$. It is clear that

$$Q_n^{(\alpha,\beta)}(y)=\sum_{j=0}^{n}\binom{n}{j}\frac{(n+\alpha+\beta+1)_j}{(\alpha+1)_j}\frac{1}{h_n^j(\alpha,\beta)}y^j.$$

Thus, the products $h_n(\alpha,\beta)(x_{n,k}(\alpha,\beta)-1)/2$ are decreasing functions of β if and only if, for any sufficiently small $\epsilon>0$, the polynomials $Q_n^{(\alpha,\beta)}(y)$ e $Q_n^{(\alpha,\beta+\epsilon)}(y)$ form a positive pair. This is equivalent to the fact that $\widetilde{Q}_n^{(\alpha,\beta+\epsilon)}(y)$ and $\widetilde{Q}_n^{(\alpha,\beta)}(y)$ form a positive pair, where $\widetilde{Q}_n^{(\alpha,\beta)}(y)=y^nQ_n^{(\alpha,\beta)}(1/y)$ denotes the inverse of $Q_n^{(\alpha,\beta)}(y)$. Let $\widetilde{H}_n(Q_n;\alpha,\beta,\epsilon)$ be the Hurwitz matrix associated with the polynomial $\widetilde{Q}_n^{(\alpha,\beta+\epsilon)}(y^2)+y\widetilde{Q}_n^{(\alpha,\beta)}(y^2)$,

$$\widetilde{H}_n(Q_n;\alpha,\beta,\epsilon)=\begin{pmatrix} 1 & \binom{n}{1}\frac{(n+\alpha+\beta+\epsilon+1)_1}{(\alpha+1)_1h_n(\alpha,\beta+\epsilon)} & \binom{n}{2}\frac{(n+\alpha+\beta+\epsilon+1)_2}{(\alpha+1)_2h_n^2(\alpha,\beta+\varepsilon)} & \cdots \\ 1 & \binom{n}{1}\frac{(n+\alpha+\beta+1)_1}{(\alpha+1)_1h_n(\alpha,\beta)} & \binom{n}{2}\frac{(n+\alpha+\beta+1)_2}{(\alpha+1)_2h_n^2(\alpha,\beta)} & \cdots \\ 0 & 1 & \binom{n}{1}\frac{(n+\alpha+\beta+\epsilon+1)_1}{(\alpha+1)_1h_n(\alpha,\beta+\epsilon)} & \cdots \\ 0 & 1 & \binom{n}{1}\frac{(n+\alpha+\beta+1)_1}{(\alpha+1)_1h_n(\alpha,\beta)} & \cdots \\ . & . & . & . \end{pmatrix}.$$

Theorem 2.1 (ii) in [3], applied to this situation implies that all $h_n(\alpha,\beta)(x_{n,k}(\alpha,\beta)-1)$, $k=1,\dots,n$ are decreasing functions of β if and only if all minors $\widetilde{\Delta}_j(Q_n;\alpha,\beta,\epsilon)$, $j=1,2,\dots,2n+1$, of $\widetilde{H}_n(Q_n;\alpha,\beta,\epsilon)$ are positive for any sufficiently small positive ϵ. On the other hand, observe that $\widetilde{\Delta}_2(Q_n;\alpha,\beta,\epsilon)$ is positive if and only if

$$(n+\alpha+\beta+1)\{h_n(\alpha,\beta+\epsilon)-h_n(\alpha,\beta)\}-\epsilon h_n(\alpha,\beta)>0,$$

which is equivalent to

$$\frac{1}{h_n(\alpha,\beta)}\frac{h_n(\alpha,\beta+\epsilon)-h_n(\alpha,\beta)}{\epsilon}>\frac{1}{n+\alpha+\beta+1}.$$

Letting ϵ tend to zero, we obtain the first statement of Theorem 2.

It shows that, if h_n is an extremal function, then h_n must be linear with respect to β. So, we set $h_n(\alpha,\beta)=\beta+n+(\alpha+1)/2+d$, $d=d_n(\alpha)$. Substituting this expression in the above Hurwitz matrix and calculating its third principal minor $\widetilde{\Delta}_3=\widetilde{\Delta}_3(Q_n;\alpha,\beta,\epsilon)$, we obtain

$$\widetilde{\Delta}_3=\frac{2n}{(\alpha+1)^2}\frac{\epsilon\{4A(n,\alpha,\epsilon,d)\beta^2+2B(n,\alpha,\epsilon,d)\beta+C(n,\alpha,\epsilon,d)\}}{(\alpha+2)(2n+\alpha+2\beta+2d+1)^2(2n+\alpha+2\beta+2\epsilon+2d+1)^2},$$

where $A(n,\alpha,\epsilon,d) = (1+\alpha)(2+\alpha) - 2d(1+n+\alpha)$. The explicit forms of the coefficients B and C of the binomial in the above numerator are pretty involved and omit them. Since the denominator of the quotient that represents $\widetilde{\Delta}_3$ is obviously positive, then this minor is positive for all sufficiently large values of β when $A(n,\alpha,\epsilon,d)$ is positive. However, this is equivalent to the inequality

$$d < \frac{(\alpha+1)(\alpha+2)}{2(n+\alpha+1)}.$$

This completes the proof of Theorem 2.

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