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ZEROS OF GEGENBAUER AND HERMITE POLYNOMIALS AND CONNECTION COEFFICIENTS

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Abstract. In this paper, sharp upper limit for the zeros of the ultraspherical polynomials are obtained via a result of Obrechkoff and certain explicit connection coefficients for these polynomials. As a consequence, sharp bounds for the zeros of the Hermite polynomials are obtained.

1. Introduction

Let $C_n^\lambda(x),\ n=0,1,\ldots,\ \lambda>-1/2$, be the ultraspherical (Gegenbauer) polynomials, orthogonal in $(-1,1)$ with respect to the weight function $(1-x^2)^{\lambda-1/2}$. Denote by $x_{nk}(\lambda),\ k=1,\ldots,n$, the zeros of $C_n^\lambda(x)$ enumerated in decreasing order, $1>x_{n1}(\lambda)>x_{n2}(\lambda)>\cdots>x_{nn}(\lambda)>-1$. The behaviour of $x_{nk}(\lambda)$ has been of interest because of their nice electrostatic interpretation and of their important role as nodes of Gaussian quadrature formulae, and vice–versa, these applications motivated further the interest on describing this behaviour more thoroughly. For instance, the fact that the positive zeros of $C_n^\lambda(x)$ decrease when λ increases is intuitively clear from the electrostatic interpretation of $x_{nk}(\lambda),\ \ k=1,\ldots,n$, as the positions of equilibrium of n unit charges in $(-1,1)$ in the field generated by two charges located at -1 and 1 whose common value is $\lambda/2+1/4$ [26, pp. 140–142]. Here the charges are distributed along infinite wires perpendicular to the interval $[-1,1]$ and because of that the field obeys the law of the logarithmic potential. On the other hand, the fact that $x_{nk}(\lambda)$ are nodes of a Gaussian quadrature formula requires sharp limits for these zeros to be established.

Since the zeros $x_{nk}(\lambda)$ are symmetric with respect to the origin, it suffices to find such limits only for the positive zeros. There have been many contributions in this direction and we refer to Chapter 6 of Szegő's classical reading [26] and to a recent survey of Elbert [7] for exhaustive number of inequalities. Generally speaking, when $\lambda\in[0,1]$, the use of Sturm's comparison theorem provides very precise bounds. This method yields (see Theorems 6.3.2 and 6.3.4 in [26])

$$\cos\left(\frac{j_k(\lambda-1/2)}{n+\lambda}\right)<x_{nk}(\lambda)<\cos\left(\frac{k-(1-\lambda)/2}{n+\lambda}\pi\right),\quad k=1,\ldots,[n/2],$$

where $0<j_1(\nu)<j_2(\nu)<\ldots$ denote the positive zeros of the Bessel function $J_\nu(x)$.

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Somehow surprisingly, most of the limits for $x_{nk}(\lambda)$, when $\lambda > 0$, were obtained only during the last two decades. Since these results are interesting mainly when λ is large enough, usually a good test for the sharpness of the corresponding bound is its behaviour when λ diverges. It is known that, for any fixed n and k, $1 \leq k \leq n$, the limit relation

$$\sqrt{\lambda} x_{nk}(\lambda) \longrightarrow h_{nk} \quad \text{as} \quad \lambda \to \infty \tag{1.1}$$

holds, where $h_{nk},\ k = 1, \ldots, n$, are the zeros of the Hermite polynomial $H_n(x)$, also arranged in decreasing order. Elbert and Laforgia [9] established a more precise asymptotic result concerning the behaviour of $x_{nk}(\lambda)$. They proved that

$$x_{nk}(\lambda) = h_{nk}\lambda^{-1/2} - \frac{h_{nk}}{8}\left(2n - 1 + 2h_{nk}^2\right)\lambda^{-3/2} + O(\lambda^{-5/2}), \quad \lambda \to \infty. \tag{1.2}$$

Moreover, it is known that, for $k = 1, \ldots, [n/2]$, the products $\sqrt{\lambda} x_{nk}(\lambda)$ tend to their horizontal asymptotes in such a way that they increase monotonically. Elbert and Siafarikas [11] proved that $[\lambda + (2n^2 + 1)/(4n + 2)]^{1/2} x_{nk}(\lambda),\ k = 1, \ldots, [n/2]$, are increasing functions of λ, for $\lambda > -1/2$, thus extending earlier results obtained in [22, 1, 17, 3], and proving a conjecture which was posed by Laforgia [23], and by Ismail, Letessier and Askey [20, 19]. Recently, the sharpness of the result of Elbert and Siafarikas was established in [5]. Also, by using the Sturm comparison theorem Gatteschi [14] obtained upper and lower bounds for the zeros of Jacobi polynomials.

To the best of our knowledge, the best bounds which hold for all the positive zeros of $C_n^\lambda(x)$ and for every $\lambda > 0$, were obtained by Elbert and Laforgia [8] through the Sturm theorem. As pointed out by Elbert [7], the result obtained in [8], implies

$$x_{nk}(\lambda) \leq \frac{\sqrt{n^2 + 2(n-1)\lambda - 1}}{n + \lambda} \cos\frac{(k-1)\pi}{n-1}, \qquad k = 1, \ldots, [n/2]. \tag{1.3}$$

For $k = 1$ the latter reduces immediately to the inequality

$$x_{n1}(\lambda) \leq \frac{\sqrt{n^2 + 2(n-1)\lambda - 1}}{n + \lambda} \tag{1.4}$$

for the largest zero of $C_n^\lambda(x)$. Limits similar to (1.4) were obtained earlier by Ifantis and Siafarikas [16, 18] and by Förster and Petras [13]. Observe that the asymptotic formula (1.2) immediately yields that

$$h_{nk} \leq \sqrt{2n-2}\cos\frac{(k-1)\pi}{n-1}, \quad k = 1, \ldots, [n/2] \tag{1.5}$$

and, in particular,

$$h_{n1} \leq \sqrt{2n-2}. \tag{1.6}$$

In what follows we adopt the following criteria for sharpness of the upper bounds for $x_{nk}(\lambda)$. The better such a limit is said to be, the better it behaves when λ diverges. Equivalently, the good upper bounds for the zeros of $C_n^\lambda(x)$ will be considered those which provide good bounds for h_{nk} through the limit relation (1.1).

The paper is organized as follows: in the next section we use a result of Ismail and Li [21] in order to establish upper bounds for $x_{n1}(\lambda)$ and h_{n1} which are better than the one appearing in (1.4) and (1.6). Section 3 contains information about the basic ingredients of our approach, namely, a theorem of Obrechkoff and a new explicit form of the connection coefficients between ultraspherical polynomials with shifted argument and the Chebyshev polynomials. These results allow us to obtain an upper estimate for the positive zeros of the ultraspherical polynomials in terms of the smallest zeros of certain Jacobi polynomials. To the best of our knowledge such a relation appears for the first time in the literature and because of that it is of interest in itself. Moreover, it provides very sharp upper limits for the zeros $x_{nk}(\lambda)$, especially when k is small in comparison with n and λ is large. These limits are obtained in Section 4. Naturally, as a consequence, we obtain precise bounds for the positive zeros of the Hermite polynomials. In Section 5 we provide numerical results and comparisons between the known limits and the bounds obtained in this paper.

2. Sharper bounds for $x_{n1}(\lambda)$ and h_{n1}

Let $\{p_k(x)\}_{k=0}^{\infty}$ be a sequence of orthonormal polynomials generated by the recurrence relation

$$\begin{aligned} p_{-1}(x) &= 0 \\ p_0(x) &= 1 \\ xp_k(x) &= a_{k+1}\ p_{k+1}(x) + b_k\ p_k(x) + a_k\ p_{k-1}(x), \end{aligned} \tag{2.1}$$

where $a_k, b_k \in \mathbb{R}$, $a_k > 0$. Then, it is well known and easy to see that the zeros of $p_n(x)$ coincide with the eigenvalues of the associated $n \times n$ Jacobi matrix

$$J_n = \begin{pmatrix} b_0 & a_1 & & & \\ a_1 & b_1 & a_2 & & \\ & a_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \\ & & & a_{n-1} & b_{n-1} \end{pmatrix}.$$

Ismail and Li [21] used a characterization of the positive definite Jacobi matrices in terms of chain sequences, due to Wall and Wetzel [28], and an ingenuous argument in order to prove the following result:

Theorem 2.1. *If the sequence of orthogonal polynomials $\{p_k(x)\}$ is defined by (2.1), then the zeros x_k, $k = 1, \dots, n$, of $p_n(x)$ belong to the interval which contains all the zeros of all the equations*

$$(x - b_{j-1})(x - b_j) = 4\ a_j^2\ \cos^2(\pi/(n+1)), \quad j = 1, \dots, n-1. \tag{2.2}$$

A simple investigation of the zeros of the equations (2.2) immediately yields:

Corollary 2.2. *If $\{b_j\}_{j=0}^{n-1}$ is a decreasing sequence and $\{a_j\}_{j=1}^{n-1}$ is an increasing one, then the smallest zero x_n of $p_n(x)$ satisfies the inequality*

$$x_n \geq \frac{1}{2}\left\{b_{n-2} + b_{n-1} - \sqrt{(b_{n-2} - b_{n-1})^2 + 16a_{n-1}^2 \cos^2(\pi/(n+1))}\right\}.$$

In particular, if $\{p_k(x)\}$ is a sequence of symmetric orthonormal polynomials, i.e. if $b_j = 0$ for $j = 1, \dots, n-1$, and $\{a_j\}_{j=1}^{n-1}$ is an increasing sequence, then the zeros x_k, $k = 1, \dots, n$, of $p_n(x)$ satisfy the inequality

$$|x_k| \leq 2a_{n-1} \cos(\pi/(n+1)).$$

Using the second statement of this corollary and the fact that for the ultraspherical polynomials $b_j = 0$,

$$a_j = \frac{1}{2}\sqrt{\frac{j(j+2\lambda-1)}{(j+\lambda-1)(j+\lambda)}},$$

and that

$$\frac{\partial a_j}{\partial j} = \frac{1}{8a_j}\frac{\lambda(\lambda-1)(2j+2\lambda-1)}{(j+\lambda-1)^2(j+\lambda)^2} > 0 \text{ for } j \geq 1 \text{ and } \lambda \geq 1,$$

we obtain the following result:

Corollary 2.3. *For any $n \geq 2$ and for every $\lambda \geq 1$ the inequality*

$$x_{n1}(\lambda) \leq \sqrt{\frac{(n-1)(n+2\lambda-2)}{(n+\lambda-2)(n+\lambda-1)}}\cos(\pi/(n+1)) \tag{2.3}$$

holds.

For the largest zero of $H_n(x)$ we have

$$h_{n1} \leq \sqrt{2n-2}\ \cos(\pi/(n+1)). \tag{2.4}$$

Observe that the bound (2.3) is better than (1.4) in the sense we compare these bounds, namely, that the corresponding limit (2.4) for h_{n1} is slightly sharper than (1.6). However, Corollary 2.3 is still a result only for the largest and not for *each* positive zero of $C_n^\lambda(x)$ and of $H_n(x)$.

3. Obrechkoff's theorem and a connection problem

In this section we obtain an upper bound for the positive zeros of the Gegenbauer polynomials in terms of the smallest zeros of certain Jacobi polynomials. Formally, this result is formulated in the statement of Theorem 3.6 and the basic tool in its proof is a beautiful theorem of Obrechkoff. In order to formulate the latter we need a definition.

Definition 3.1. The finite sequence of functions $f_1, \dots, f_n$ obeys Descartes' rule of signs in the interval (a, b) if the number of zeros in (a, b), where the multiple zeros are counted with their multiplicities, of any real linear combination

$$\alpha_1 f_1(x) + \ldots + \alpha_n f_n(x)$$

does not exceed the number of sign changes in the sequence $\alpha_1, \dots, \alpha_n$.

Theorem 3.2 (Obrechkoff [24])**.** *Let the sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ be defined by the recurrence relation*

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \quad n \geq 0, \tag{3.1}$$

where $a_n, b_n, c_n \in \mathbb{R}$, $a_n, c_n > 0$. If ζ_n denotes the largest zero of $p_n(x)$, then the sequence of polynomials $p_0, \dots, p_n$ obeys Descartes' rule of signs in (ζ_n, ∞).

Favard's theorem [12], [2, Theorem 4.4], implies that the requirements of the Theorem 3.2 are equivalent to the requirement that $\{p_n\}$ is a sequence of orthogonal polynomials. If we denote by $Z(f; (a, b))$ the number of the zeros, counting their multiplicities, of the function $f(x)$ in (a, b), and by $S(\alpha_1, \dots, \alpha_n)$ the number of sign changes in the sequence $\alpha_1, \dots, \alpha_n$, Obrechkoff's theorem can be reformulated in the following more succinct form.

Corollary 3.3. *Let the orthogonal polynomials $p_k(x)$, $k = 0, 1, \ldots, n$ be normalized in such a way that their leading coefficients are all of the same sign and let ζ_n be the largest zero of $p_n(x)$. Then, for any sequence $\alpha_0, \ldots, \alpha_n$, which is not identically zero,*

$$Z(\alpha_0 p_0(x) + \cdots + \alpha_n p_n(x); (\zeta_n, \infty)) \leq S(\alpha_0, \ldots, \alpha_n).$$

We must emphasize that it is essential that the polynomials are normalized in such a way that their leading coefficients are positive.

Some history about Theorem 3.2 as well as some application to zeros of orthogonal polynomials were discussed in [4]. There we employed the connection problems for the classical orthogonal polynomials. In this paper we obtain more precise results using the explicit solution of the following more sophisticated connection problem.

Theorem 3.4. *The connection coefficients $A_{n,m}(s)$ in the expansion*

$$C_n^{\lambda}(s\,x) = \sum_{m=0}^{n} A_{n,m}(s)\, T_m(x) \quad s \neq 0, \tag{3.2}$$

where $C_n^{(\lambda)}(x)$ are the monic Gegenbauer polynomials and $T_m(x)$ are the monic Chebyshev polynomials of the first kind, are given by $A_{n,m}(s) \equiv 0$ when $n-m$ is odd and by

(3.3)
$$\begin{aligned} A_{n,m}(s) \; &= \; \frac{(-1)^{(n-m)/2}}{m!((n-m)/2)!}\, \frac{s^m}{2^{3n-m}}\, \frac{(2\lambda)_n\,(n+2\lambda)_n\, n!}{(\lambda+1/2)_n\,(\lambda+(n+m)/2)_{(n-m)/2}\,(\lambda)_n} \\ &\times \; {}_2F_1\left(\begin{array}{c} (m-n)/2, (2\lambda+m+n)/2 \\ m+1 \end{array} \,\middle|\, s^2 \right), \end{aligned}$$

when $n-m$ is even, where $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$ denotes the Pochhammer symbol.

Proof. In order to derive the explicit formulae for $A_{n,m}(s)$ we use the method known as the `Navima` algorithm [15, 25]. The main feature of this method is to obtain a recurrence relation for the connection coefficients. This recurrence relation is in fact a difference equation which can be solved in many cases. Formally, this procedure goes as follows for our problem. Consider the differential operator

$$\mathcal{D}_{n,s} := \left(1 - s^2\,x^2\right) \frac{\mathrm{d}^2}{\mathrm{d}x^2} - (1+2\,\lambda)\; s^2\, x \frac{\mathrm{d}}{\mathrm{d}x} + n\;(2\,\lambda+n)\; s^2\, \mathcal{I}$$

where $\mathcal{I}$ stands for the identity operator. Since $\mathcal{D}_{n,s}\left[C_n^{\lambda}(s\,x)\right] = 0, \;\; n = 1, 2, \ldots,$ then the application of $\mathcal{D}_{n,s}$ to both sides of (3.2) yields

$$0 = \sum_{m=0}^{n} A_{n,m}(s)\, \mathcal{D}_{n,s}\left[T_m(x)\right]. \tag{3.4}$$

On using the recurrence relation for the Chebyshev polynomials and the identity $4(m+1)T_m(x) = 4T'_{m+1}(x) - T'_{m-1}(x)$ we reduce (3.4) to the sum

$$0 = \sum_{m=0}^{n} A_{n,m}(s) \left\{ \left(\frac{(n-m)\,(2\,\lambda+m+n-1)\,s^2}{(1+m)\,(2+m)} \right) T''_{m+2}(x) \right.$$
$$+ \left(1 - \frac{(m\,(2+m) + 2\,\lambda\,(2+n) + (-3+n)\,(2+n))\,s^2}{2\,m\,(2+m)} \right) T''_m(x)$$
$$\left. + \left(\frac{(2\,\lambda - m + n - 3)\,(2+m+n)\,s^2}{16\,m\,(1+m)} \right) T''_{m-2}(x) \right\},$$

where $T''_m(x) := 0$ when m is an index smaller than 2. Rewriting the latter as a combination of the linearly independent polynomials $T''_m(x)$, $m = 2, \ldots, n+2$, we obtain the desired three–term recurrence relation for the connection coefficients:

$$\begin{aligned} (3.5)\quad & (m-1)(n-m+2\lambda-2)(n+m+2)s^2 A_{n,m+2}(s) \\ & - 8m\left(2(1-m^2) + (n^2+m^2+2(n+1)\lambda-2)s^2\right) A_{n,m}(s) \\ & - 16(m+1)(m-n-2)(n+m+2\lambda-2)s^2 A_{n,m-2}(s) = 0\,, \end{aligned}$$

which holds for $2 \le m \le n$, with the initial conditions $A_{n,n}(s) = s^n$ for every nonnegative integer n, $A_{n,n-1}(s) = 0$ for all $n \in \mathbb{N}$ and $A_{n,m}(s) = 0$ for each $m > n$. We have to prove that the expressions given on the right–hand side of (3.3) satisfy the recurrence relation (3.5). Thus, we need to show that the identity

$$\begin{aligned} (3.6)\quad & -4(m+2)(m+1)^2 m(m-1) f_{n,m-2}(s) \\ & - 2(m+2)(m+1)m\left(2(1-m^2) + (n^2+m^2+2(n+1)\lambda-2)s^2\right) f_{n,m}(s) \\ & - \frac{1}{2}(m-1)(n-m)(n+m+2)\left(\frac{n+m}{2}+\lambda\right)(n-m-2-2\lambda) f_{n,m+2}(s) = 0\,, \end{aligned}$$

holds for the hypergeometric polynomials

$$f_{n,m}(s) = {}_2F_1\left(\begin{matrix} (m-n)/2, (2\lambda+m+n)/2 \\ m+1 \end{matrix} \,\middle|\, s^2 \right)$$

whenever $n - m$ is an even integer. Using the Maclaurin expansion of the these polynomials, we express the polynomial on the left–hand side of (3.6) in the form

$$\sum_{j=0}^{(n-m+2)/2} b_j s^{2j}.$$

Performing this lengthly but straightforward procedure we obtain the following general expression for the coefficient b_j:

$$\begin{aligned} b_j = & \frac{((m-n)/2+1)_{j-2}((n+m)/2+\lambda+1)_{j-2}(m-n)(m+n+2\lambda)}{4j!(m+3)_{k-2}} \times \\ & \{-(m+1)(m+j)(m+j-1)(m-n-2)(m+n-2+2\lambda) \\ & -(m+1)m(m-1)(m-n+2j-2)(m+n+2\lambda+2j-2) \\ & -2m(m+j)j(n^2+m^2-2+2(n+1)\lambda) \\ & +(m-1)(n+m+2)(n-m-2+2\lambda)j(j-1)\} \end{aligned}$$

and it can be verified that this expression is identically zero.

It is worth mentioning that a different argument can be used to justify the fact that $f_{n,m}(s)$ are solutions of (3.6). Very recently Vidūnas [27] presented an algorithm for computing general contiguous relations for ${}_2F_1$ hypergeometric series, which in particular yields a relation of the form

$$Q(1,1,2)\,{}_2F_1\left(\begin{array}{c} a-1\,,\,b-1 \\ c-2 \end{array}\,\middle|\, z\right) - Q(-1,-1,-2)\,{}_2F_1\left(\begin{array}{c} a+1\,,\,b+1 \\ c+2 \end{array}\,\middle|\, z\right)$$
$$= [P(-1,-1,-2)Q(1,1,2) - P(1,1,2)Q(-1,-1,-2)]\;{}_2F_1\left(\begin{array}{c} a\,,\,b \\ c \end{array}\,\middle|\, z\right),$$

where

$$\begin{aligned} Q(1,1,2) &= \frac{c^2\,(1+c)\,(1-z)}{b\,(b-c)\,(c-a)}\,, \\ P(1,1,2) &= \frac{c\,(1+c)\,(c+b\,z-c\,z)}{b\,(a-c)\,(b-c)}\,, \\ Q(-1,-1,-2) &= \frac{a(1-z)z^2}{c-1}\,, \\ P(-1,-1,-2) &= \frac{(c-a-1)\,(c+z-b\,z-2)}{(c-2)\,(c-1)}\,. \end{aligned}$$

□

Now we prove two important properties of the zeros of $A_{n,m}(s)$ defined by (3.3).

Lemma 3.5. *Let $n-m$ be a nonnegative even integer. Then $A_{n,m}(s)$ is an algebraic polynomial of degree n with positive leading coefficient. The origin is a zero of $A_{n,m}(s)$ with multiplicity m and the remaining $n-m$ zeros are real, distinct, symmetric with respect to the origin and belong to $(-1,1)$.*

Moreover, if $n-m$ and $n-m-2$ are two consecutive nonnegative even integers, then the positive zeros of $A_{n,m}(s)$ and $A_{n,m+2}(s)$ interlace.

Proof. Recall the hypergeometric representation of the monic Jacobi polynomials

$$P_N^{(\alpha,\beta)}(x) := \frac{2^N(\alpha+1)_N}{(N+\alpha+\beta+1)_N}\;{}_2F_1\left(\begin{array}{c} -N, N+\alpha+\beta+1 \\ \alpha+1 \end{array}\,\middle|\, \frac{1-x}{2}\right),$$

orthogonal on $(-1,1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ when $\alpha,\beta>-1$. Set

$$(m-n)/2 = -N, \qquad m+1 = \alpha+1, \qquad (2\lambda+m+n)/2 = N+\alpha+\beta+1.$$

Then $N=(n-m)/2$, $\alpha=m$ and $\beta=\lambda-1$ and we see that $A_{n,m}(s)$ is a multiple of a Jacobi polynomial of degree $(n-m)/2$. More precisely

$$A_{n,m}(s) = (-1)^{(n-m)/2} c_{n,m}\, s^m\, P_{(n-m)/2}^{(m,\lambda-1)}(1-2s^2), \quad \lambda>0, \tag{3.7}$$

with

$$c_{n,m} = 2^{3(m-n)/2}\frac{n!}{((n-m)/2)!\,((n+m)/2)!}\,.$$

This establishes the first statement of the lemma.

The second one is a consequence of Sturm's comparison theorem [26, Theorem 1.82.1]. Equation (4.24.1) in [26] shows that

$$u(x) = (1-x)^{(m+1)/2}(1+x)^{\lambda/2}P_{(n-m)/2}^{(m,\lambda-1)}(x)$$

is a solution of the second–order differential equation

$$u''(x) + F_{n,m,\lambda}(x)u(x) = 0,$$

where

$$F_{n,m,\lambda}(x) = \frac{1}{4}\left(\frac{1-m^2}{(1-x)^2} + \frac{1-(\lambda-1)^2}{(1+x)^2} + \frac{n^2-m^2+2(n+1)\lambda}{1-x^2}\right).$$

Since

$$F_{n,m,\lambda}(x) - F_{n,m+2,\lambda}(x) = \frac{2(m+1)}{(1-x)^2(1+x)},$$

then $F_{n,m,\lambda}(x) > F_{n,m+2,\lambda}(x)$ for any $x \in (-1,1)$ and Sturm's theorem immediately implies that the zeros of $P^{(m,\lambda-1)}_{(n-m)/2}(x)$ and of $P^{(m+2,\lambda-1)}_{(n-m-2)/2}(x)$ interlace, which itself yields that the positive zeros of $A_{n,m}(s)$ and $A_{n,m+2}(s)$ interlace. □

In what follows we suppose that the zeros $x_{nk}(\alpha,\beta)$, $k=1,\dots,n$, of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ are arranged in decreasing order.

Theorem 3.6. *Let $n \in \mathbb{N}$ and $\varepsilon = n (\mathrm{mod}\, 2)$, i.e. $\varepsilon = 0$ if n is even, and $\varepsilon = 1$ if n is odd. Then the inequalities*

$$x_{nk}(\lambda) \leq \sqrt{\frac{1 - x_{(n-\varepsilon)/2-k+1,(n-\varepsilon)/2-k+1}(\varepsilon+2k-2,\lambda-1)}{2}}\cos\frac{\pi}{2n} \tag{3.8}$$

hold for every k, $k = 1,\dots,[n/2]$, and $\lambda > 0$.

Proof. For any real $s \neq 0$ the zeros of $C_n^\lambda(sx)$ are $s^{-1}x_{nk}(\lambda)$. Since the largest zero of the Chebyshev polynomial of degree n is $\cos(\pi/2n)$, then Obrechkoff's result, as stated in Corollary 3.3, and Theorem 3.4 imply immediately that

$$Z(C_n^\lambda(sx); (\cos(\pi/2n),\infty)) \leq S(A_{n,0}(s),\dots,A_{n,n}(s)).$$

Therefore, for any fixed $s \neq 0$ and $\lambda > -1/2$,

$$\#\{k \ :\ s^{-1}x_{nk}(\lambda) > \cos(\pi/2n)\} \leq S(A_{n,0}(s),\dots,A_{n,n}(s)),$$

where $\#D$ denotes the number of the elements of the finite set D. Equivalently,

$$x_{nk}(\lambda) \leq s_k \cos(\pi/2n) \tag{3.9}$$

provided $S(A_{n,0}(s_k),\dots,A_{n,n}(s_k)) \leq k-1$. Thus we need to count the number of sign changes in the sequence $A_{n,0}(s_k),\dots,A_{n,n}(s_k)$. By Lemma 3.5, $A_{n,j}(s)$ are identically zero when $n-j$ is odd, and when $n-j$ is even these polynomials obey three important properties:

- They have have positive leading coefficients;
- $A_{n,j}(s)$ has exactly $(n-j)/2$ positive zeros.
- The positive zeros of two consecutive polynomials $A_{n,j}(s)$ and $A_{n,j+2}(s)$ strictly interlace.

Therefore $S(A_{n,0}(s),\dots,A_{n,n}(s))$ may change only when s passes through a zeros of either of the polynomials $A_{n,n}(s), A_{n,n-2}(s),\dots,A_{n,\varepsilon}(s)$.

Then the above properties immediately imply that $S(A_{n,0}(s),\dots,A_{n,n}(s)) = 0$ if $s \geq s_1$, where s_1 is the largest zero of $A_{n,\varepsilon}(x)$, which itself coincides with $((1-x_{(n-\varepsilon)/2,(n-\varepsilon)/2}(\varepsilon,\lambda-1))/2)^{1/2}$, $\lambda > 0$, because of the relation (3.7) between $A_{n,j}$ and the Jacobi polynomials. This yields (3.8) for $k=1$.

If s is not less than the largest zero s_2 of $A_{n,\varepsilon+2}(x)$, then

$$S(A_{n,0}(s),\dots,A_{n,n}(s)) \leq 1\,.$$

Since

$$s_2 = ((1 - x_{(n-\varepsilon-2)/2,(n-\varepsilon-2)/2}(\varepsilon+2, \lambda-1))/2)^{1/2}$$

then (3.9) yields the inequality (3.8) for $k = 2$.

Similarly, if k is any index with $1 \le k \le [n/2]$, the interlacing property of the zeros of the "connecting polynomials" $A_{n,j}(s)$, implies that

$$S(A_{n,0}(s_k), \ldots, A_{n,n}(s_k)) \le k - 1$$

if s_k is not less than the largest zero of $A_{n,\varepsilon+2k-2}$. Again, having in mind relation (3.7), we obtain

$$s_k = \sqrt{\frac{1 - x_{(n-\varepsilon)/2-k+1,(n-\varepsilon)/2-k+1}(\varepsilon+2k-2, \lambda-1)}{2}}$$

and this proves inequality (3.8). □

4. Upper limits for the positive zeros of Gegenbauer and Hermite polynomials

In this Section we establish the main results of the paper. In fact, these are immediate consequences of Theorem 3.6 and some known results we apply related to some sharp limits for the extreme zeros of the Jacobi polynomials. However, it turns out that the bounds that we obtain for the positive zeros of Gegenbauer and Hermite polynomials are very sharp.

Theorem 4.1. *For any $n \ge 2$, and for every $\lambda > 1/2$ the inequality*

$$\begin{aligned}(4.1)\quad x_{nk}^2(\lambda) \le \frac{\cos^2(\pi/(2n))}{2(n+\lambda)^2} \quad &\{2 + n^2 + 2n\lambda + 4k^2 + \lambda - 2k(3-2\varepsilon) - \varepsilon(3-\varepsilon) \\ &+ \sqrt{(n-2k+2-\varepsilon)(n+2k-1+\varepsilon)(n-2k+2\lambda+1-\varepsilon)(n+2k+2\lambda-2+\varepsilon)}\}\end{aligned}$$

holds.

Proof. Elbert, Laforgia and Rodonó [10] established lower and upper limits for the zeros of the Jacobi polynomials. In particular, they proved that the inequality

$$x_{NN}(\alpha,\beta) \ge \frac{(\beta-\alpha)(\alpha+\beta+1) - 4\sqrt{N(N+\alpha+\frac{1}{2})(N+\beta+\frac{1}{2})(N+\alpha+\beta+1)}}{(2N+\alpha+\beta+1)^2}$$

for the smallest zero of $P_N^{(\alpha,\beta)}(x)$ holds for $\alpha, \beta > -1/2$. Employing Theorem 3.6 and the latter inequality for $N = (n-\varepsilon)/2 - k + 1$, $\alpha = \varepsilon + 2k - 2$ and $\beta = \lambda - 1$, and performing some straightforward calculations we obtain (4.1). □

As a consequence of (1.1) we obtain:

Corollary 4.2. *For every $n \in \mathbb{N}$ and each k, $k = 1, \ldots, [n/2]$ the inequality*

$$(4.2)\qquad h_{nk} \le \sqrt{n + \frac{1}{2} + \sqrt{(n+2k+\varepsilon-1)(n-2k-\varepsilon+2)}}\ \cos\frac{\pi}{2n}$$

for the positive zero h_{nk} of $H_n(x)$ holds. In particular,

$$h_{nk} \leq \sqrt{n+\frac{1}{2}+\sqrt{(n+2k-1)(n-2k+2)}}\ \cos\frac{\pi}{2n} \quad \text{if } n \text{ is even,}$$
$$h_{nk} \leq \sqrt{n+\frac{1}{2}+\sqrt{(n+2k)(n-2k+1)}}\ \cos\frac{\pi}{2n} \quad \text{if } n \text{ is even.}$$

Thus, the largest zero h_{n1} of $H_n(x)$ satisfies the inequality

$$h_{n1} \leq \sqrt{n+\frac{1}{2}+\sqrt{n(n+1)}}\ \cos\frac{\pi}{2n} \quad \text{if } n \text{ is even,} \tag{4.3}$$

$$h_{n1} \leq \sqrt{n+\frac{1}{2}+\sqrt{(n-1)(n+2)}}\ \cos\frac{\pi}{2n} \quad \text{if } n \text{ is odd.} \tag{4.4}$$

In what follows we obtain new sharp bounds for the zeros of $C_n^\lambda(x)$ and $H_n(x)$.

Theorem 4.3. *For every $n \in \mathbb{N}$, for each k, $k = 1, 2, \ldots, [n/2]$, and for every $\lambda > 2k-1+\varepsilon$, the following bound for the zero $x_{nk}(\lambda)$ of Gegenbauer polynomial $C_n^\lambda(x)$ holds:*

(4.5)

$$\begin{aligned}
x_{nk}^2(\lambda) \leq \frac{1}{2}\cos^2\left(\frac{\pi}{2\,n}\right) &\left\{1+\frac{(2\,k-\lambda+\varepsilon-1)\,(2\,k+\lambda+\varepsilon-3)}{(\lambda+n-5)\,(\lambda+n-1)}+\frac{1}{(\lambda+n-3)}\right.\\
\times &\left(\frac{4\,(1-2\,k+\lambda-\varepsilon)^2\,(2\,k+\lambda+\varepsilon-3)^2}{(\lambda+n-5)^2\,(\lambda+n-1)^2}+\cos^2\left(\frac{2\pi}{4-2k+n-\varepsilon}\right)(2k-n+\varepsilon)\right.\\
&\left.\left.\times\ \frac{(2+2\,k-2\,\lambda-n+\varepsilon)\,(2\,k+n+\varepsilon-4)\,(2\,(-3+k+\lambda)+n+\varepsilon)}{(\lambda+n-4)\,(\lambda+n-2)}\right)^{1/2}\right\}.
\end{aligned}$$

Proof. For the orthonormal Jacobi polynomials we have

$$\begin{aligned}
b_j = b_j(\alpha,\beta) &= \frac{\beta^2-\alpha^2}{(2j+\alpha+\beta+2)(2j+\alpha+\beta)},\\
a_j = a_j(\alpha,\beta) &= \frac{2}{2j+\alpha+\beta}\sqrt{\frac{j(j+\alpha+\beta)(j+\alpha)(j+\beta)}{(2j+\alpha+\beta-1)(2j+\alpha+\beta+1)}}.
\end{aligned}$$

Obviously, if $\beta^2 > \alpha^2$, then $\{b_j\}_{j=0}^{n-1}$ is a decreasing sequence. In order to investigate the behaviour of $\{a_j\}_{j=1}^{n-1}$, observe that

$$\frac{\partial a_j^2}{\partial j} = \frac{4A(j,\alpha,\beta)}{(2j+\alpha+\beta-1)^2(2j+\alpha+\beta)^3(2j+\alpha+\beta+1)^2},$$

with

$$\begin{aligned}
A(j,\alpha,\beta) = {}& 4(2\alpha^2+2\beta^2-1)j^4+8(\alpha+\beta)(2\alpha^2+2\beta^2-1)j^3\\
&+2(\alpha+\beta)^2(5\alpha^2+5\beta^2+2\alpha\beta-3)j^2+2(\alpha+\beta)^3((\alpha+\beta)^2-1)j\\
&+\alpha\beta(\alpha+\beta)^2((\alpha+\beta)^2-1).
\end{aligned}$$

The half–plane $\alpha+\beta \leq 1$ contains both the disc $2\alpha^2+2\beta^2 \leq 1$ and the ellipse $5\alpha^2+5\beta^2+2\alpha\beta \leq 3$. Thus the requirement $\alpha+\beta \geq 1$ already guarantees that the coefficients of j^4, j^3 and j^2 of $A(j,\alpha,\beta)$ are nonnegative. Since j is a positive integer, then for the last two terms we have

$$2(\alpha+\beta)^3((\alpha+\beta)^2-1)j+\alpha\beta(\alpha+\beta)^2((\alpha+\beta)^2-1)\geq \\ (\alpha+\beta+1)(\alpha+\beta-1)(\alpha+\beta)^2\left\{(\alpha+1)(\beta+1)+(\alpha+\beta-1)\right\}.$$

Hence, the sum of the coefficient of j and the free term in $A(j,\alpha,\beta)$ is always nonnegative whenever $\alpha,\beta>-1$ and $\alpha+\beta\geq 1$. Therefore the requirements that $\{b_j\}$ is decreasing and $\{a_j\}$ is increasing are satisfied simultaneously if $\alpha,\beta>-1$, $\alpha+\beta\geq 1$ and $\beta\geq\alpha$. These observations, together with the statement of Corollary 2.2, imply that

$$x_{NN}(\alpha,\beta)\geq\frac{1}{2}\left\{b_{N-2}+b_{N-1}-\sqrt{(b_{N-2}-b_{N-1})^2+16a_{N-1}^2\cos^2(\pi/(N+1))}\right\},$$

provided $\alpha,\beta>-1$, $\alpha+\beta\geq 1$ and $\beta\geq\alpha$. This immediately yields that, if $b_j=b_j(\varepsilon+2k-2,\lambda-1)$ and $a_j=a_j(\varepsilon+2k-2,\lambda-1)$, then

$$x_{(n-\varepsilon)/2-k+1,(n-\varepsilon)/2-k+1}(\varepsilon+2k-2,\lambda-1) \\ \geq\frac{1}{2}\left\{b_{N-2}+b_{N-1}-\sqrt{(b_{N-2}-b_{N-1})^2+16a_{N-1}^2\cos^2(\pi/(N+1))}\right\},$$

with $N=(n-\varepsilon)/2-k+1$, if $\lambda+\varepsilon+2k-3\geq 0$ and $\lambda\geq\varepsilon+2k-1$. Then, bearing in mind the latter estimate and applying Theorem 3.6 we obtain (4.5). □

Again the limit relation (1.1) immediately yields:

Corollary 4.4. *For every $n\in\mathbb{N}$ and each k, $k=1,\dots,[n/2]$ the inequality*

(4.6)
$$h_{nk}\leq\sqrt{n-2+\sqrt{1+(n-2k-\varepsilon)(n+2k-4+\varepsilon)\cos^2\frac{2\pi}{n-2k+4-\varepsilon}}}\ \cos\frac{\pi}{2n}$$

for the positive zero h_{nk} of $H_n(x)$ holds. In particular,

$$h_{nk}\leq\sqrt{n-2+\sqrt{1+(n-2k)(n+2k-4)\cos^2\frac{2\pi}{n-2k+4}}}\ \cos\frac{\pi}{2n}$$

for even n, and

$$h_{nk}\leq\sqrt{n-2+\sqrt{1+(n-2k-1)(n+2k-3)\cos^2\frac{2\pi}{n-2k+3}}}\ \cos\frac{\pi}{2n}$$

for odd n. Thus, the largest zero h_{n1} of $H_n(x)$ satisfies the inequality

$$h_{n1}\leq\sqrt{n-2+\sqrt{1+(n-2)^2\cos^2\frac{2\pi}{n+2}}}\ \cos\frac{\pi}{2n}\qquad\text{if } n \text{ is even,}\tag{4.7}$$

$$h_{n1}\leq\sqrt{n-2+\sqrt{1+(n-1)(n-3)\cos^2\frac{2\pi}{n+1}}}\ \cos\frac{\pi}{2n}\qquad\text{if } n \text{ is odd.}\tag{4.8}$$

5. Numerical experiments

In the present section we compare the upper bounds for the zeros of $C_n^\lambda(x)$ and $H_n(x)$, obtained in this paper, with the limits known in the literature. First of all we emphasize that the limits (1.3) and (1.5) are the best known when k is close to $[n/2]$, i.e. when we are interested in bounds for the small positive zeros of the polynomials under discussion. This is so because the argument of the cosine in these formulae is close to $\pi/2$.

The results obtained in this paper provide the best known upper bounds for the large zeros of $C_n^\lambda(x)$ and $H_n(x)$, i.e. for small k and especially for the largest zeros $x_{n1}(\lambda)$ and h_{n1}.

As it was already mentioned, our principal result which yield the remaining one almost immediately, is Theorem 3.6. It is somehow surprising how sharp inequality (3.8) is. In Table 1 we show the numerical values of the two sides of (3.8) for several values of n, k and λ.

Table 1. Gegenbauer zeros

n	k	λ	$x_{nk}(\lambda)$	Th. 3.6
10	1	0.1	0.98501	0.98565
20	1	1/2	0.99312	0.99399
50	1	2	0.99627	0.99679
15	1	2000	0.10019	0.10169
20	2	1/2	0.96397	0.99396
50	2	2	0.98898	0.99679
50	3	2	0.97808	0.99677

Recall that we are interested mainly in estimates for $x_{nk}(\lambda)$ when λ is large enough and because of that we adopted the criteria for comparison of the estimates through the corresponding estimates for the zeros of $H_n(x)$.

In what follows we shall show some numerical evidences that the upper limits (4.7) and (4.8) are the best known and, because of that, we may consider the result in Theorem 4.3 for $k = 1$ the best upper limit for the zeros of the Gegenbauer polynomials, for large values of λ.

The graphs included in Figure 1 show the difference between the upper limit obtained in (2.4), for even n, and in (4.3), as well as the difference between the right–hand side of in (2.4), for odd n, and in (4.4). Both differences are shown as functions of n. It is worth emphasizing that more extensive numerical experiments show that these differences tend to zero monotonically decreasing and because of that remain positive. The graphs show that the limit for h_{n1}, given in (2.4), is still better than those obtained in Corollary 4.2.

However, it turns out that the bounds for h_{n1}, as given in (4.7) and (4.8), are sharper than that in (2.4). This is shown by the graphs included in Figure 2. They present the differences between the bound given in (2.4), for even n, and the right–hand side of (4.7), as well as the similar difference between the limit in (2.4), for odd n, and the right–hand side of (4.8). Again, these differences remain positive for all positive integers n.

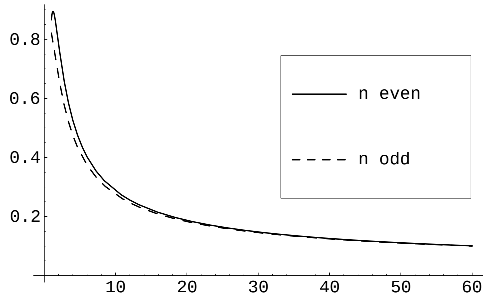

FIGURE 1

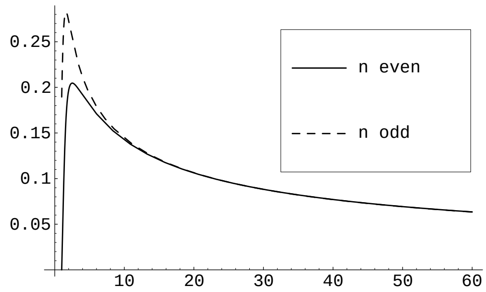

FIGURE 2

Finally, let us compare (4.7) and (4.8) with the best upper bound given in Szegő's classical book [26]. The bound which appears in [26, (6.32.6) and (6.32.7)] is obtained using the powerful Sturm comparison theorem, it is given in terms of the first zero i_1 of the Airy function, as defined in Section 1.81 of [26], and it reads as

$$(5.1)\quad h_{n1} \leq (2n+1)^{1/2} - 6^{-1/3} i_1 (2n+1)^{-1/2} \approx (2n+1)^{1/2} - 1.85575(2n+1)^{-1/2}\,.$$

The two final graphs, included in Figure 3, contain the differences of this limit, for even and odd n, respectively, and the one obtained in (4.7) and (4.8), as functions of n. Thus, our Corollary 4.4 provides better limits for h_{n1} than (5.1).

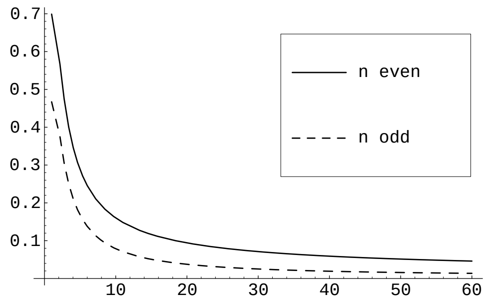

FIGURE 3

Table 2 contains numerical evidences of the sharpness of the limits for h_{n1}, given by Corollary 4.4, compared to those provided by Szegő's estimate (5.1), by Corollary 2.3, and by Corollary 4.2.

TABLE 2. Hermite zeros

n	h_{n1}	Cor. 4.4	Eq. (5.1)	Eq. (2.4)	Cor. 4.2
3	1.22474	1.22474	1.94434	1.41421	2.23533
10	3.43615	3.82530	4.17762	4.07078	4.52487
20	5.38748	5.92305	6.11330	6.09556	6.38291
30	6.86334	7.43829	7.57264	7.57669	7.79928
50	9.18240	9.77579	9.86522	9.88072	10.0448
100	13.40648	13.99181	14.04655	14.06444	14.17565

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