Archives of Inequalities and Applications xx (200x)xxx-xxx

LANDAU AND KOLMOGOROFF TYPE POLYNOMIAL INEQUALITIES II

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ABSTRACT. Let $0 < j < m \leq n$. Kolmogoroff type inequalities of the form

$$\|f^{(j)}\|^2 \leq A\|f^{(m)}\|^2 + B\|f\|^2$$

which hold for algebraic polynomials of degree n are established. Here the norm is defined by $\int f^2(x)d\mu(x)$, where $d\mu(x)$ is any distribution associated with the Jacobi, Laguerre or Bessel orthogonal polynomials. In particular we characterize completely the positive constants A and B, for which the Landau weighted polynomial inequalities

$$\|f'\|^2 \leq A\|f''\|^2 + B\|f\|^2$$

hold. For some special values of A and B this second result reduces to a Stein type of inequality obtained by Agarwal and Milovanović [1].

AMS (MOS) Subject Classification. Primary 41A17, 26D15; Secondary 33C45.

Keywords. Landau and Kolmogoroff type inequalities, Markov's inequality, Jacobi polynomials, Laguerre polynomials, Bessel polynomials, extremal polynomials, Rayleigh-Ritz theorem.

1. INTRODUCTION AND STATEMENT OF RESULTS

A classical result of Landau [7] states that, for every $f \in C^2[0,1]$, the inequality $\|f'\| \leq 4$ holds provided $\|f\| = 1$ and $\|f''\| = 4$, where $\|.\|$ denotes the uniform norm in $[0,1]$. Kolmogoroff [6] generalized this result establishing inequalities of the form

$$\|f^{(j)}\| \ \leq \ K(m,j)\, \|f^{(m)}\|^{\frac{j}{m}}\, \|f\|^{1-\frac{j}{m}}, \ \ 0 < j < m, \ f \in C^m[0,1],$$

with the best possible constants $K(m,j)$.

Denote by π_n the space of real algebraic polynomials of degree not exceeding n. In what follows we suppose $0 < j < m \leq n$.

In the recent paper [2], Kolmogoroff type inequalities

$$\|f^{(j)}\|^2 \leq A\|f^{(m)}\|^2 + B\|f\|^2, \quad f \in \pi_n,$$

were obtained for various values of the constants A and B, where the norm was defined by $\|f\|^2 = \int_{-\infty}^{\infty} f^2(x)\exp(-x^2)dx$. Moreover, complete characterization of the positive constants A and B, for which the corresponding Landau type polynomial inequalities

$$\|f'\|^2 \leq A\|f''\|^2 + B\|f\|^2,$$

hold, was given. Thus the principal inequalities obtained in [2] generalize previous results of Varma [13] and Bojanov and Varma [4].

The research is supported by the Brazilian Science Foundations FAPESP under Grants 00/07856-8, 99/12054-9 and 97/6280-0, and CNPq under Grant 300645/95-3.

The main purpose of this paper is to extend further the result of [2] to various weighted Landau and Kolmogoroff type polynomial inequalities. The norms under consideration are

$$\|f\|^2_{(0,J)} := \|f\|^2_{(\alpha,\beta)} = \int_{-1}^{1} f^2(x)(1-x)^\alpha(1+x)^\beta dx, \quad \alpha > -1,\ \beta > -1, \tag{1.1}$$

$$\|f\|^2_{(0,L)} := \|f\|^2_{(\alpha)} = \int_0^\infty f^2(x)x^\alpha e^{-x}dx, \quad \alpha > -1, \tag{1.2}$$

and

$$\|f\|^2_{(0,B)} := \|f\|^2_{(\alpha,\pi_n)} = \int_0^\infty f^2(x)x^{\alpha-2}e^{-\beta/x}dx. \tag{1.3}$$

Recall that the Jacobi polynomials

$$p_n^{(J)}(x) := P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2})$$

and the Laguerre polynomials

$$p_n^{(L)}(x) := L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n; \alpha+1; x)$$

are orthogonal with respect to the inner products which generate the norms (1.1) and (1.2). Recently, Srivastava [12] proved that the generalized Bessel polynomials

$$p_n^{(B)}(x) = y_n(x;\alpha,\beta) = {}_2F_0(-n, \alpha+n-1; -; -x/\beta)$$

obey the orthogonal property

$$\int_0^\infty x^{\alpha-2}e^{-\beta/x}y_r(x;\alpha,\beta)y_s(x;\alpha,\beta)dx = \beta^{\alpha-1}\frac{r!}{1-\alpha-2r}\Gamma(2-\alpha-r)\delta_{rs}, \tag{1.4}$$

$Re(\alpha) < 1-s-r,\ Re(\beta) > 0,\ r,s \in I\!N_0 := I\!N \cup \{0\}$.

For $\alpha,\beta \in I\!R$, $\beta > 0$, $n \in I\!N_0$ and $2n < 1-\alpha$, the norm (1.3) is well-defined in the space π_n. Moreover, the above property (1.4) yields that $\{y_k(x;\alpha,\beta)\}_{k=0}^n$ are orthogonal with respect to the inner product that generates the norm (1.3). Under the same restriction on α, β and n, $\|p^{(i)}\|_{(\alpha+2i,\pi_{n-1})}$ is well-defined for every $p \in \pi_n$ since the inequality $2n < 1-\alpha$ is equivalent to $2(n-i) < 1-(\alpha+2i)$. As it was pointed out in [3],

$$y_n^{(i)}(x;\alpha,\beta) = \frac{(-1)^i}{\beta^i}\frac{n!}{(n-i)!}\frac{\Gamma(2-\alpha-n)}{\Gamma(2-\alpha-n-i)}y_n(x;\alpha+2i,\beta) \tag{1.5}$$

and the latter polynomial is orthogonal with respect to the inner product which generates

$$\|f\|_{(i,B)} := \|f\|_{(\alpha+2i,\pi_{n-i})}.$$

It is known also that for any positive integers i the polynomials $P_n^{(\alpha+i,\beta+i)}(x)$ and $L_n^{(\alpha+i)}(x)$ are orthogonal with respect to the inner products which generate the norms

$$\|f\|_{(i,J)} := \|f\|_{(\alpha+i,\beta+i)}$$

and

$$\|f\|_{(i,L)} := \|f\|_{(\alpha+i)},$$

respectively (see formulae (4.21.7) and (5.1.14) in Szegő [11]).

In order to formulate more succinctly our Kolmogoroff type polynomial inequalities, we need to introduce some additional denotations. When an inequality concerns the norms $\|\,.\,\|_{(0,J)}$, $\|\,.\,\|_{(j,J)}$ and $\|\,.\,\|_{(m,J)}$ and the indexed constants θ_k, μ_k and S_k appear, we shall mean the following values:

$$\begin{aligned} \theta_k &= \frac{(k-j)!}{(k-m)!}\frac{\Gamma(k+m+\alpha+\beta+1)}{\Gamma(k+j+\alpha+\beta+1)}, \quad k=m,\dots,n, \\ \mu_k &= \frac{(k-j)!}{k!}\frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(k+j+\alpha+\beta+1)}, \quad k=j,\dots,n, \\ S_k &= \frac{\mu_k-\mu_{k+1}}{\theta_{k+1}-\theta_k} \\ &= \frac{j(k-m+1)!}{(m-j)(k+1)!}\frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(k+m+\alpha+\beta+1)}, \quad k=m-1,\dots,n-1. \end{aligned} \tag{1.6}$$

In the inequalities associated with the weighted norms $\|\,.\,\|_{(0,L)}$, $\|\,.\,\|_{(j,L)}$ and $\|\,.\,\|_{(m,L)}$, the constants are understood to be

$$\begin{aligned} \theta_k &= \frac{(k-j)!}{(k-m)!}, \quad k=m,\dots,n, \\ \mu_k &= \frac{(k-j)!}{k!}, \quad k=j,\dots,n, \\ S_k &= \frac{j(k-m+1)!}{(m-j)(k+1)!}, \quad k=m-1,\dots,n-1. \end{aligned} \tag{1.7}$$

Finally, the values of θ_k, μ_k and S_k in the Kolmogoroff inequalities involving the norms $\|\,.\,\|_{(0,B)}$, $\|\,.\,\|_{(j,B)}$ and $\|\,.\,\|_{(m,B)}$, are

$$\begin{aligned} \theta_k &= \frac{(k-j)!}{(k-m)!}\frac{\Gamma(2-\alpha-k-j)}{\Gamma(2-\alpha-k-m)}, \quad k=m,\dots,n, \\ \mu_k &= \frac{(k-j)!}{k!}\frac{\Gamma(2-\alpha-k-j)}{\Gamma(2-\alpha-k)}, \quad k=j,\dots,n, \\ S_k &= \frac{j(k-m+1)!}{(m-j)(k+1)!}\frac{\Gamma(2-\alpha-k-m)}{\Gamma(2-\alpha-k)}, \quad k=m-1,\dots,n-1. \end{aligned} \tag{1.8}$$

Furthermore, in all the cases we set $\theta_k = 0,\ k = j,\dots,m-1$.

Thus, we formulate our main Kolmogoroff type weighted polynomial inequalities:

Theorem 2.1. Let $j < m \le n$ be positive integers and D positive constant.

(i) If $D \le S_{n-1}$, then

$$\|f^{(j)}\|^2_{(j,\nu)} \le \frac{1}{D\theta_n+\mu_n}\left\{D\|f^{(m)}\|^2_{(m,\nu)} + \|f\|^2_{(0,\nu)}\right\},\ \nu = J, L, B, \tag{1.9}$$

for every $f \in \pi_n$. Moreover, equality is attained if and only if $f(x)$ is a constant multiple of $p_n^{(\nu)}(x),\ \nu = J, L, B$.

(ii) If $S_m < D < S_{m-1}$, then

$$\|f^{(j)}\|^2_{(j,\nu)} \le \frac{1}{D\theta_m+\mu_m}\left\{D\|f^{(m)}\|^2_{(m,\nu)} + \|f\|^2_{(0,\nu)}\right\},\ \nu = J, L, B, \tag{1.10}$$

for every $f \in \pi_n$. Moreover, equality is attained if and only if $f(x) = cp_m^{(\nu)}(x),\ \nu = J, L, B$, where c is a constant.

(iii) If $D > S_{m-1}$, then

$$\|f^{(j)}\|^2_{(j,\nu)} \le \frac{1}{\mu_{m-1}}\left\{D\|f^{(m)}\|^2_{(m,\nu)} + \|f\|^2_{(0,\nu)}\right\},\ \nu = J, L, B, \tag{1.11}$$

for every $f \in \pi_n$. Moreover, equality is attained if and only if $f(x)$ is a constant multiple of $p_{m-1}^{(\nu)}(x),\ \nu = J, L, B$.

(iv) If $D = S_{m-1}$, then the inequalities (1.10) and (1.11) coincide and they hold for every $f \in \pi_n$. In this case equality is attained if and only if $f(x) = c_1 p_m^{(\nu)}(x) + c_2 p_{m-1}^{(\nu)}(x),\ \nu = J, L, B$, where c_1 and c_2 are constants.

In the case $j = 1$ and $m = 2$ we provide a complete characterization of the positive constant D for which the corresponding Landau type polynomial inequalities hold. In this case, we suppose that the constants θ_k, μ_k and R_k are defined as follows. In the Landau inequalities for the norm associated with the Jacobi polynomials, they are

$$\begin{aligned} \theta_k &= (k-1)(k+\alpha+\beta+2), \quad k = 1, \dots, n, \\ \mu_k &= [k(k+\alpha+\beta+1)]^{-1}, \quad k = 1, \dots, n, \\ R_k &= [k(k-1)(k+\alpha+\beta)(k+\alpha+\beta+1)]^{-1}, \quad k = 2, \dots, n. \end{aligned} \tag{1.12}$$

In the inequalities corresponding to the Laguerre weighted function we set:

$$\begin{aligned} \theta_k &= (k-1), \quad k = 1, \dots, n, \\ \mu_k &= k^{-1}, \quad k = 1, \dots, n, \\ R_k &= [k(k-1)]^{-1}, \quad k = 2, \dots, n, \end{aligned} \tag{1.13}$$

and in the Bessel case

$$\begin{aligned} \theta_k &= (k-1)(-\alpha-k), \quad k = 1, \dots, n, \\ \mu_k &= [k(1-\alpha-k)]^{-1}, \quad k = 1, \dots, n, \\ R_2 &= [2(-\alpha)(-\alpha-1)]^{-1}, \\ R_n &= [2n(1-\alpha-n)(-\alpha-1)]^{-1}. \end{aligned} \tag{1.14}$$

Observe that the new definitions of θ_k and μ_k are not confusing to the values given previously. Indeed, θ_k and μ_k in (1.12), (1.13) and (1.14) are obtained by setting $j = 1$ and $m = 2$ in the correspondind formulae (1.6), (1.7) and (1.8).

Theorem 2.2. Let D be a positive constant.

(i) If $0 < D \le R_n$ then

$$\|f'\|^2_{(1,\nu)} \le \frac{1}{D\theta_n + \mu_n} \left\{ D\|f''\|^2_{(2,\nu)} + \|f\|^2_{(0,\nu)} \right\}, \ \nu = J, L, B, \tag{1.15}$$

for every $f \in \pi_n$. Moreover, equality is attained if and only if $f(x)$ is a constant multiple of $p_n^{(\nu)}(x),\ \nu = J, L, B$.

(ii) a) If $R_{k+1} < D < R_k$, where $k \in I\!N,\ 2 \le k \le n-1$, then

$$\|f'\|^2_{(1,\nu)} \le \frac{1}{D\theta_k + \mu_k} \left\{ D\|f''\|^2_{(2,\nu)} + \|f\|^2_{(0,\nu)} \right\}, \ \nu = J, L, \tag{1.16}$$

for every $f \in \pi_n$. Moreover, equality is attained if and only if $f(x) = c p_k^{(\nu)}(x),\ \nu = J, L$, where c is a constant.

b) If $R_n < D < R_2$, then

$$\|f'\|^2_{(1,B)} \le \frac{1}{D\theta_2 + \mu_2} \left\{ D\|f''\|^2_{(2,B)} + \|f\|^2_{(0,B)} \right\}, \tag{1.17}$$

for every $f \in \pi_n$. Moreover, equality is attained if and only if $f(x) = cp_2^{(B)}(x)$, where c is a constant.

(iii) If $R_2 < D < \infty$, then

$$\|f'\|^2_{(1,\nu)} \le \frac{1}{\mu_1}\left\{D\|f''\|^2_{(2,\nu)} + \|f\|^2_{(0,\nu)}\right\},\ \nu = J, L, B, \tag{1.18}$$

for every $f \in \pi_n$. Moreover, equality is attained if and only if $f(x)$ is a constant multiple of $p_1^{(\nu)}(x),\ \nu = J, L, B$.

(iv) If $D = S_k$, for some integer k, then the inequalities

$$\|f'\|^2_{(1,\nu)} \le \frac{1}{D\theta_k+\mu_k}\left\{D\|f''\|^2_{(2,\nu)} + \|f\|^2_{(0,\nu)}\right\} \tag{1.19}$$

$$,\quad \nu = J, L,$$

$$\|f'\|^2_{(1,\nu)} \le \frac{1}{D\theta_{k+1}+\mu_{k+1}}\left\{D\|f^{(\prime\prime}\|^2_{(2,\nu)} + \|f\|^2_{(0,\nu)}\right\} \tag{1.20}$$

coincide and they hold for every $f \in \pi_n$. In this case equality is attained if and only if $f(x) = d_1p_k^{(\nu)}(x) + d_2p_{k+1}^{(\nu)}(x),\ \nu = J, L$, where d_1 and d_2 are constants.

Setting $D = [n(n+\alpha+\beta+1)]^{-2}$ in Theorem 2.2. (i) we obtain, for $\nu = J$, the inequality

$$\begin{aligned}\|f'\|^2_{(1,J)} \le\ & \frac{1}{(2n-1)(\alpha+\beta)+2(n^2+n-1)}\|f''\|^2_{(2,J)} \\ & + \frac{n^2(n+\alpha+\beta+1)^2}{(2n-1)(\alpha+\beta)+2(n^2+n-1)}\|f\|^2_{(0,J)},\quad f \in \pi_n,\end{aligned}$$

where equality is attained if and only if $f(x)$ is a constant multiple of $p_n^{(J)}(x)$.

If, in Theorem 2.2 (i), we set $D = n^{-2}$ and $\nu = L$ we obtain the inequality

$$\|f'\|^2_{(1,L)} \le \frac{1}{(2n-1)}\|f''\|^2_{(2,L)} + \frac{n^2}{(2n-1)}\|f\|^2_{(0,L)},\quad f \in \pi_n,$$

with equality if and only if $f(x) = cp_n^{(L)}(x)$, where c is a constant. These are exactly the result of the Theorem 1.10.4 in [10] for the Jacobi and Laguerre polynomials, respectively.

2. PROOFS OF THE THEOREMS

Following the ideas given in [2], for each $\nu = J, L, B$ our objective is to study the extremal problem

$$F(C_1, C_2) = \min\left\{\frac{C_1\|f^{(m)}\|^2_{(m,\nu)} + C_2\|f\|^2_{(0,\nu)}}{\|f^{(j)}\|^2_{(j,\nu)}} : f \in \pi_n, f(x) \ne 0\right\},$$

for any given integers j, m, n, $0 < j < m \le n$, and positive constants C_1 and C_2.

For this purpose let the sequences $\{\gamma_i\}_{i=j}^n$ be defined by

$$\begin{aligned}\gamma_i &= C_2\mu_i, & i &= j, \dots, m-1,\\ \gamma_i &= C_1\theta_i + C_2\mu_i, & i &= m, \dots, n.\end{aligned} \tag{2.1}$$

where $\{\mu_i\}_{i=j}^n$ and $\{\theta_i\}_{i=m}^n$ are given in (1.6), (1.7) and (1.8) for $\nu = J, L, B$ respectively.

We need also some basic properties of the Jacobi and Laguerre polynomials:

$$\begin{aligned}\int_{-1}^1 p_i^{(J)}(x)p_k^{(J)}(x)w_{(J)}(x)dx\ =\ & \frac{\Gamma(\alpha+i+1)\Gamma(\beta+i+1)}{\Gamma(\alpha+\beta+i+1)} \\ & \times\ \frac{2^{\alpha+\beta+1}}{(2i+\alpha+\beta+1)i!}\delta_{i,k},\end{aligned} \tag{2.2}$$

where $w_{(J)}(x) = (1-x)^\alpha (1+x)^\beta$, and $\alpha,\ \beta > -1$,

$$\int_0^\infty L_i^{(\alpha)}(x) L_k^{(\alpha)}(x) x^\alpha e^{-x} dx = \frac{\Gamma(\alpha+i+1)}{i!}\delta_{i,k},\ \alpha > -1, \tag{2.3}$$

$$\frac{d}{dx} P_k^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+k+1}{2} P_{k-1}^{(\alpha+1,\beta+1)}(x), \tag{2.4}$$

$$\frac{d}{dx} L_k^{(\alpha)}(x) = -L_{k-1}^{(\alpha+1)}(x). \tag{2.5}$$

The identities (2.2), (2.3), (2.4) and (2.5) correspond to (4.3.3), (5.1.1), (4.2.7) and (5.1.14) in [11].

Consider now the following normalization of the orthogonal polynomials $p_i^{(\nu)}(x)$, $\nu = J, L, B$:

$$\tilde{p}_i^{(\nu)}(x) = c_i p_i^{(\nu)}(x),$$

where the constants $c_i, i = 0, \ldots, n$, are defined by: $c_i = 1,\ i = 0, \ldots, j-1$, and for $i = j, \ldots, n$

$$c_i = \left\{ \frac{2^{\alpha+\beta+1}}{(2i+\alpha+\beta+1)(i-j)!} \frac{\Gamma(\alpha+i+1)\Gamma(\beta+i+1)\Gamma(\alpha+\beta+i+j+1)}{[\Gamma(\alpha+\beta+i+1)]^2} \right\}^{-1/2},$$

if $\nu = J$,

$$c_i = \left\{ \frac{\Gamma(\alpha+i+1)}{(i-j)!} \right\}^{-1/2}$$

when $\nu = L$ and, in the Bessel case ($\nu = B$),

$$c_i = \left\{ \frac{(i!)^2}{(i-j)!} \frac{\beta^{\alpha-1}}{(-\alpha-2i+1)} \frac{\Gamma^2(-\alpha-i+2)}{\Gamma(-\alpha-i-j+2)} \right\}^{-1/2}.$$

Obviously, every $f \in \pi_n$ can be uniquely represented in the form

$$f(x) = \sum_{k=0}^n a_k \tilde{p}_k^{(\nu)}(x), \quad \nu = J, L, B.$$

¿From the relations (2.2), (2.3), (1.4) and the definition of the polynomials $\tilde{p}_k^{(\nu)}(x)$, $\nu = J, L, B$, we have

$$\|f\|^2_{(0,\nu)} = \sum_{k=0}^n \mu_k a_k^2, \quad \nu = J, L, B,$$

where μ_k, $k = j, \ldots, n$, are defined in (1.6), (1.7), (1.8), respectively for $\nu = J, L, B$.

For $k = 0, \ldots, j-1$, the constants μ_k are given by

$$\mu_k = \frac{2^{\alpha+\beta+1}}{(2k+\alpha+\beta+1)k!} \frac{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}{\Gamma(\alpha+\beta+k+1)} \quad \text{for } \nu = J,$$

$$\mu_k = \frac{\Gamma(\alpha+k+1)}{k!}$$

for the Laguerre case and, if $\nu = B$,

$$\mu_k = \frac{k!}{1-\alpha-2k}\Gamma(2-\alpha-k)\beta^{\alpha-1}.$$

Again using the relations (2.2), (2.3), (1.4), from (2.4), (2.5), (1.5) and the definition of the polynomials $\tilde{p}_k^{(\nu)}(x)$, we obtain

$$\|f^{(j)}\|_{(j,\nu)} = \sum_{k=j}^n a_k^2 c_k^2 \|(p_k^{(\nu)})^{(j)}\|_{(j,\nu)} = \sum_{k=j}^n a_k^2, \quad \nu = J, L, B.$$

In the same way, we obtain

$$\|f^{(m)}\|_{(m,\nu)} = \sum_{k=m}^{n} \theta_k a_k^2, \quad \nu = J, L, B,$$

where θ_k, $k = m, \dots, n$, are defined in the equations (1.6), (1.7), (1.8), respectively for $\nu = J, L, B$.

Thus,

$$F(C_1, C_2) = \min \left\{ \frac{\sum_{k=0}^{m-1} C_2 \mu_k a_k^2 + \sum_{k=m}^{n} (C_1 \theta_k + C_2 \mu_k) a_k^2}{\sum_{k=j}^{n} a_k^2} \;:\; a_0, \dots, a_n \in I\!R \right\}.$$

Obviously, the above minimum is attained for $a_0 = \dots = a_{j-1} = 0$. Hence, we have to determine the minimum of $a^t A a$, subject to $a^t a = 1$, where A is the diagonal matrix

$$diag\,(C_2\mu_j, \dots, C_2\mu_{m-1}, C_1\theta_m + C_2\mu_m, \dots, C_1\theta_n + C_2\mu_n)\,.$$

By the Rayleigh-Ritz Theorem (Theorem 4.2.2 on page 176 in [5]), our problem reduces to determine the smallest eigenvalue of A. In summary, for each $\nu = J, L, B$, we have proven:

Lemma 2.3. For any given integers $j < m \le n$ positive constants C_1 and C_2

$$F(C_1, C_2) = \gamma_k := \min\{\gamma_j, \dots, \gamma_n\}, \tag{2.6}$$

where γ_k, $k = j, \dots, n$, are given in the equations (2.1). Moreover, the extremal polynomials for which the minimum is attained is a constant multiple of $p_k^{(\nu)}(x)$.

Then, to prove Theorem 2.1, we need to analyze the behaviour of the sequences $\{\gamma_k\}_{k=j}^n$ for each $\nu = J, L, B$. For this purpose, consider the following results.

Lemma 2.4. The sequences $\{\mu_k\}_{k=j}^n$, defined in the equations (1.6), (1.7) and (1.8), respectively for $\nu = J, L, B$, are decreasing.

Proof. i) Consider $\nu = J$. Then, by (1.6) we have

$$\begin{aligned}
\frac{\mu_k}{\mu_{k+1}} &= \frac{(k-j)!}{k!} \frac{\Gamma(\alpha+\beta+k+1)}{\Gamma(\alpha+\beta+k+j+1)} \frac{(k+1)!}{(k+1-j)!} \frac{\Gamma(\alpha+\beta+k+j+2)}{\Gamma(\alpha+\beta+k+2)} \\
&= \frac{k+1}{k+1-j} \frac{\alpha+\beta+k+j+1}{\alpha+\beta+k+1} \\
&= \left(1 + \frac{j}{k-j+1}\right)\left(1 + \frac{j}{\alpha+\beta+k+1}\right) > 1, \ \alpha, \beta > -1.
\end{aligned}$$

ii) If $\nu = L$, by (1.7)

$$\mu_k - \mu_{k+1} = \frac{(k-j)!}{k!} - \frac{(k-j+1)!}{(k+1)!} = j\,\frac{(k-j)!}{(k+1)!} > 0, \ \alpha > -1.$$

iii) Finally, for $\nu = B$, equation (1.8) gives us

$$\begin{aligned}
\frac{\mu_k}{\mu_{k+1}} &= \frac{(k-j)!}{k!} \frac{\Gamma(2-\alpha-k-j)}{\Gamma(2-\alpha-k)} \frac{(k+1)!}{(k+1-j)!} \frac{\Gamma(1-\alpha-k-j)}{\Gamma(1-\alpha-k)} \\
&= \frac{k+1}{k+1-j} \frac{1-\alpha-k-j}{1-\alpha-k} > 1,
\end{aligned}$$

since $2n < 1 - \alpha$ and, consequently, $(k+1-j)(1-\alpha-k) < (k+1)(1-\alpha-k-j)$. ■

In the same manner we prove

Lemma 2.5. The sequences $\{\theta_k\}_{k=m}^n$, defined in the equations (1.6), (1.7) and (1.8), respectively for $\nu = J, L, B$, are increasing.

Proof of Theorem 2.1 As the sequences $\{\mu_k\}_{k=j}^{m-1}$ is decreasing for $\nu = J, L, B$, then the smaller among the numbers $C_2\mu_j, \ldots, C_2\mu_{m-1}$ is $C_2\mu_{m-1}$. Thus, according to Lemma 2.3, we need to find the smaller among $\gamma_m, \ldots, \gamma_n$ and to compare with $C_2\mu_{m-1}$.
Now, consider the monotonicy of the sequences $\{\gamma_k\}_{k=m}^n$ for $\nu = J, L, B$. Since

$$\gamma_{k+1} - \gamma_k = C_1(\theta_{k+1} - \theta_k) + C_2(\mu_k - \mu_{k+1}),$$

then: *a*) the sequences $\{\gamma_k\}_{k=m}^n$ are increasing if $D := C_1/C_2 \geq S_k := (\mu_k - \mu_{k+1})/(\theta_{k+1} - \theta_k)$ for $k = m, \ldots, n-1$ and *b*) $\{\gamma_k\}_{k=m}^n$ are decreasing if $D \leq S_k$ for $k = m, \ldots, n-1$.
Straightforward calculations show that for $\nu = J, L, B$, S_k, $k = m, \ldots, n-1$, are given, respectively, in the equations (1.6), (1.7) and (1.8). But, if $\nu = J$,

$$\frac{S_{k+1}}{S_k} = \frac{(k+2-m)(\alpha+\beta+k+1)}{(k+2)(\alpha+\beta+k+m+1)}.$$

When $\nu = L$,

$$\frac{S_{k+1}}{S_k} = \frac{k+2-m}{k+2},$$

and, in the Bessel case,

$$\frac{S_{k+1}}{S_k} = \frac{(k+2-m)(1-\alpha-k)}{(k+2)(1-\alpha-k-m)}.$$

Then, $S_{k+1}/S_k < 1$ for $\nu = J, L, B$ and $k = m, \ldots, n-1$. This means that $\{S_k\}$ are decreasing sequences. Hence, if $D \geq S_m$ then γ_k are increasing and $\gamma_m = \min\{\gamma_k,\ k = m, \ldots, n\}$. Thus, we have

$$F(C_1, C_2) = \min_{j \leq k \leq n} \gamma_k = \min\{\gamma_{m-1}, \gamma_m\},\ \nu = J, L, B.$$

Observe that $\gamma_{m-1} < \gamma_m$ if $\tilde{S} := (\mu_{m-1} - \mu_m)/\theta_m < D$ and $\gamma_{m-1} \geq \gamma_m$ otherwise. But

$$\begin{aligned}
\tilde{S} &= \frac{j}{(m-j)m!}\frac{\Gamma(\alpha+\beta+m)}{\Gamma(\alpha+\beta+2m)} \quad \text{for } \nu = J,\\
\tilde{S} &= \frac{j}{(m-j)m!} \quad \text{if } \nu = L,\\
\tilde{S} &= \frac{j}{(m-j)m!}\frac{\Gamma(3-\alpha-2m)}{\Gamma(3-\alpha-m)} \quad \text{for } \nu = B.
\end{aligned}$$

In view of the above identities we can conclude:

(1) If $D \geq S_m$ and $D > \tilde{S}$, then $\gamma_{m-1} < \gamma_m$ and $F(C_1, C_2) = \gamma_{m-1}$;

(2) If $S_m \leq D < \tilde{S}$, then $\gamma_{m-1} > \gamma_m$ and $F(C_1, C_2) = \gamma_m$.
The later cases (1) and (2) correspond to the statements (iii) and (ii) of Theorem 2.1.
The above observation b) and the monotonicy of S_k, $k = m, \ldots, n-1$, imply that the sequences $\{\gamma_k\}_{k=m}^n$ are decreasing provided $D \leq S_{n-1}$. Hence, in these cases we have $F(C_1, C_2) = \min\{\gamma_{m-1}, \gamma_n\}$, $\nu = J, L, B$. In order to compare γ_{m-1} and γ_n, note that $\gamma_{m-1} < \gamma_n$ if $\tilde{R} := (\mu_{m-1} - \mu_n)/\theta_n < D$ and $\gamma_{m-1} \geq \gamma_n$ otherwise.

In view of the identities

$$\begin{aligned}\tilde{R} &= \frac{(n-m)!}{(n-j)!}\frac{\Gamma(\alpha+\beta+n+j+1)}{\Gamma(\alpha+\beta+n+m+1)}\left[\frac{(m-j-1)!}{(m-1)!}\frac{\Gamma(\alpha+\beta+m)}{\Gamma(\alpha+\beta+m+j)}\right.\\ &\quad \left. - \frac{(n-j)!}{n!}\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+n+j+1)}\right], \ \nu = J,\\ \tilde{R} &= \frac{(n-m)!}{(n-j)!}\left[\frac{(m-j-1)!}{(m-1)!} - \frac{(n-j)!}{n!}\right], \ \nu = L,\end{aligned}$$

and

$$\begin{aligned}\tilde{R} &= \frac{(n-m)!}{(n-j)!}\frac{\Gamma(2-\alpha-n-m)}{\Gamma(2-\alpha-n-j)}\left[\frac{(m-j-1)!}{(m-1)!}\frac{\Gamma(3-\alpha-m-j)}{\Gamma(3-\alpha-m)}\right.\\ &\quad \left. - \frac{(n-j)!}{n!}\frac{\Gamma(2-\alpha-n-j)}{\Gamma(2-\alpha-n)}\right], \ \nu = B,\end{aligned}$$

we need relations between the latter expressions and D. On the other hand, the inequality $\frac{m!}{(m-j)!} < \frac{n!}{(n-j)!}$, $j < m < n$, yields $S_{n-1} < \tilde{R}$, $\nu = J, L, B$. If $D \leq S_{n-1}$ and $D < \tilde{R}$, then $\gamma_{m-1} > \gamma_n$ and $F(C_1, C_2) = \gamma_n$. This corresponds to the statement (i) of the theorem. ■

Proof of Theorem 2.2 Since $j = 1$ and $m = 2$, Lemma 2.3 shows that we need to determine

$$\min\{C_2\mu_1, C_1\theta_2 + C_2\mu_2, \ldots, C_1\theta_n + C_2\mu_n\}.$$

In order to this, in all the cases $\nu = J$, $\nu = L$ or $\nu = B$, we shall find the smaller among the numbers $C_1\theta_2 + C_2\mu_2, \ldots, C_1\theta_n + C_2\mu_n$, and we shall compare it to $C_2\mu_1$.
In what follows, up to the final observation in this proof, we shall assume that $C_1 + C_2 = 1$, $C_1, C_2 > 0$.
Consider first the case $\nu = J$. Then, by equations in (1.12), for $k = 2, \ldots, n$ we have

$$\gamma_k = C_1\theta_k + C_2\mu_k = C_2\frac{1}{k(k+\alpha+\beta+1)} + (1-C_2)(k-1)(k+\alpha+\beta+2).$$

Define the function

$$g_J(x) = C_2\frac{1}{x(x+\alpha+\beta+1)} + (1-C_2)(x-1)(x+\alpha+\beta+2) \quad \text{for} \quad 2 \leq x \leq n.$$

Since $g_J(k) = \gamma_k$ for $k = 2, \ldots, n$, then our problem reduces to investigate the behaviour of $g_J(x)$ when C_1 and C_2 belong to the segment $C_1 + C_2 = 1$, $C_1, C_2 > 0$. Note that the only zero of

$$g_J'(x) = (2x+\alpha+\beta+1)\left(1 - C_2 - \frac{C_2}{x^2(x+\alpha+\beta+1)^2}\right)$$

that can belong to the interval $[2, n]$ is

$$x = \frac{-(\alpha+\beta+1) + \left[(\alpha+\beta+1)^2 + 4\sqrt{C_2/(1-C_2)}\right]^{1/2}}{2} > 0, \ \alpha, \beta > -1.$$

But, for $x \geq 2$

$$g_J''(x) = 2(1-C_2) + \frac{2C_2}{x^3(x+\alpha+\beta+1)^3}\left[3x^2 + 3(\alpha+\beta+1)x + (\alpha+\beta+1)^2\right] > 0.$$

Hence $g_J(x)$ is convex on $[2, \infty)$ and it can attain its absolute minimum on there at $x = \{-(\alpha+\beta+1) + [(\alpha+\beta+1)^2 - 4\sqrt{C_2/(1-C_2)}]^{1/2}\}/2$. Thus, we can conclude that:

- If
$$\frac{-(\alpha+\beta+1)+\left[(\alpha+\beta+1)^2+4\sqrt{C_2/(1-C_2)}\right]^{1/2}}{2}<2,$$
then $\gamma_{\min}=\gamma_2$;
- If
$$\frac{-(\alpha+\beta+1)+\left[(\alpha+\beta+1)^2+4\sqrt{C_2/(1-C_2)}\right]^{1/2}}{2}>n,$$
then $\gamma_{\min}=\gamma_n$;
- If
$$k\le\frac{-(\alpha+\beta+1)+\left[(\alpha+\beta+1)^2+4\sqrt{C_2/(1-C_2)}\right]^{1/2}}{2}<k+1,$$
where $2\le k\le n-1$, then $\gamma_{\min}=\min\{\gamma_k,\gamma_{k+1}\}$.

In order to determine the smaller between γ_k and γ_{k+1}, observe that
$$\gamma_{k+1}<\gamma_k \ \text{ if } \ \frac{k(k+1)(k+\alpha+\beta+1)(\alpha+\beta+k+2)}{k(k+1)(k+\alpha+\beta+1)(k+\alpha+\beta+2)+1}<C_2$$
and $\gamma_{k+1}\ge\gamma_k$ otherwise. It is clear that $\gamma_k=\gamma_{k+1}$ if and only if
$$C_2=\frac{k(k+1)(k+\alpha+\beta+1)(\alpha+\beta+k+2)}{k(k+1)(k+\alpha+\beta+1)(k+\alpha+\beta+2)+1}.$$
Set
$$y:=\frac{-(\alpha+\beta+1)+\left[(\alpha+\beta+1)^2+4\sqrt{C_2/(1-C_2)}\right]^{1/2}}{2}$$
for any $C_2,\ 0<C_2<1$.
If
$$C_2=\frac{k(k+1)(k+\alpha+\beta+1)(k+\alpha+\beta+2)}{k(k+1)(k+\alpha+\beta+1)(k+\alpha+\beta+2)+1}$$
then the point of minimum of $g_J(x)$ is
$$y_k:=\frac{-(\alpha+\beta+1)+\left[(\alpha+\beta+1)^2+4\sqrt{k(k+1)(k+\alpha+\beta+1)(k+\alpha+\beta+2)}\right]^{1/2}}{2}.$$
Observe that $k<y_k<k+1$. Since the function $g_J(x)$ is convex, then $\gamma_{\min}=\gamma_{k+1}$ if and only if $y_k<y<y_{k+1}$ and this conclusion holds for $k=1,\dots,n-2$. The latter inequality is equivalent to $A_k<C_2<A_{k+1}$, where
$$A_k:=\frac{k(k+1)(k+\alpha+\beta+1)(k+\alpha+\beta+2)}{k(k+1)(k+\alpha+\beta+1)(k+\alpha+\beta+2)+1}.$$
Let us compare, in each of these cases, γ_{k+1} to $\gamma_1=C_2/(\alpha+\beta+2)$.
For $C_2\in\left(0,\dfrac{6(\alpha+\beta+3)(\alpha+\beta+4)}{6(\alpha+\beta+3)(\alpha+\beta+4)+1}\right)$, we need to compare γ_1 and γ_2. Since
$$\gamma_2-\gamma_1=(\alpha+\beta+4)\left(1-\frac{2(\alpha+\beta+2)(\alpha+\beta+3)+1}{2(\alpha+\beta+2)(\alpha+\beta+3)}\right),$$
then

- $\gamma_1<\gamma_2$ for $0<C_2<\dfrac{2(\alpha+\beta+2)(\alpha+\beta+3)}{2(\alpha+\beta+2)(\alpha+\beta+3)+1}$;
- $\gamma_1=\gamma_2$ for $C_2=\dfrac{2(\alpha+\beta+2)(\alpha+\beta+3)}{2(\alpha+\beta+2)(\alpha+\beta+3)+1}$ and
- $\gamma_2<\gamma_1$ for $\dfrac{2(\alpha+\beta+2)(\alpha+\beta+3)}{2(\alpha+\beta+2)(\alpha+\beta+3)+1}<C_2<\dfrac{6(\alpha+\beta+3)(\alpha+\beta+4)}{6(\alpha+\beta+3)(\alpha+\beta+4)+1}$.

Let now k be any integer, such that $2 \le k \le n$ and let

$$C_2 \in (A_k, A_{k+1}) =: \Delta_k.$$

Since

$$\gamma_{k+1} - \gamma_1 = k(k+\alpha+\beta+3)\left(1 - \frac{(k+1)(k+\alpha+\beta+2)(\alpha+\beta+2)+1}{(k+1)(k+\alpha+\beta+2)(\alpha+\beta+2)}C_2\right) \le 0$$

if and only if $\dfrac{(k+1)(k+\alpha+\beta+2)(\alpha+\beta+2)}{(k+1)(k+\alpha+\beta+2)(\alpha+\beta+2)+1} < C_2$ and this latter inequality always holds for $C_2 \in \Delta_k$. Then we have $\gamma_{\min} = \gamma_{k+1}$ for every $C_2 \in \Delta_k$.
Finally, we have $\gamma_n < \gamma_1$ for every

$$C_2 \in \left(\frac{n(n-1)(n+\alpha+\beta)(n+\alpha+\beta+1)}{n(n-1)(n+\alpha+\beta)(n+\alpha+\beta+1)+1}, 1\right) = \Delta_{n-1}$$

because

$$\gamma_n - \gamma_1 = (n-1)(n+\alpha+\beta+2))\left(1 - \frac{n(n+\alpha+\beta+1)(\alpha+\beta+2)+1}{n(n+\alpha+\beta+1)(\alpha+\beta+2)}C_2\right) < 0$$

is equivalent to $\dfrac{n(n+\alpha+\beta+1)(\alpha+\beta+2)}{n(n+\alpha+\beta+1)(\alpha+\beta+2)+1} < C_2$.
Recall that all considerations have been done under the restriction $C_1 + C_2 = 1$. The restrictions $C_2 \in \Delta_k$ can be easily transformed into equivalent restrictions for $D = C_1/C_2$. We omit this detail. The result is:

- If $[2(\alpha+\beta+2)(\alpha+\beta+3)]^{-1} < D < \infty$, then $\gamma_{\min} = \gamma_1$;
- If $[(k+1)(k+2)(k+\alpha+\beta+2)(k+\alpha+\beta+3)]^{-1} < D < [k(k+1)(k+\alpha+\beta+1)(k+\alpha+\beta+2)]^{-1}$, then $\gamma_{\min} = \gamma_{k+1},\ k = 1, \dots, n-2$;
- If $0 < D < [n(n-1)(n+\alpha+\beta+1)(n+\alpha+\beta+2)]^{-1}$, then $\gamma_{\min} = \gamma_n$.

This completes the proof for the Jacobi case.
The proof for the case $\nu = L$ is similar to the Hermite case in [2]. By considering the equations in (1.13) we obtain

$$\gamma_k = C_1\theta_k + C_2\mu_k = C_2\frac{1}{k} + (1-C_2)(k-1) \quad \textit{for} \quad k = 2, \dots, n.$$

Observe that it is exactly the equation for γ_k in the proof of (Theorem 2, [2]).
Finally, consider the Bessel case, that is, $\nu = B$. Then, by equations in (1.14) we have

$$\gamma_k = C_1\theta_k + C_2\mu_k = C_2\frac{1}{k(-\alpha-k+1)} + (1-C_2)(k-1)(-\alpha-k) \quad \textit{for} \quad k = 2, \dots, n.$$

In this case, let $g_B(x)$ be defined by

$$g_B(x) = C_2\frac{1}{x(-\alpha-x+1)} + (1-C_2)(x-1)(-\alpha-x), \quad 2 \le x \le n.$$

Using partial fraction decomposition, $g_B(x)$ can be written in the form

$$g_B(x) = \frac{C_2}{2-\alpha}\left(\frac{1}{x} + \frac{1}{-\alpha-x+1}\right) + (1-C_2)(x-1)(-\alpha-x).$$

Remember that $2n < 1-\alpha$ in the Bessel case. Then, $2-\alpha > 0$ and $n < (-\alpha+1)/2$. Hence, $g_B(x)$ is concave on $[2, n]$.
Since $g_B(k) = \gamma_k$, $k = 2, \dots, n$, then our problem reduces to determine the smaller between γ_2 and γ_n and to compare it to γ_1. Observe that $\gamma_n < \gamma_1$ if $[2n(-\alpha-1)(1-\alpha-n+]/[2n(-\alpha-1)(1-\alpha-n)+1] <$

C_2 and $\gamma_n \geq \gamma_1$ otherwise.
Let us compare γ_2 to $\gamma_1 = C_2/(-\alpha)$. Since

$$\gamma_2 - \gamma_1 = \frac{-\alpha - 2}{2(-\alpha)(-\alpha-1)} \{2(-\alpha)(-\alpha-1) - [2(-\alpha)(-\alpha-1)+1]\, C_2\}$$

then

- $\gamma_2 < \gamma_1$ for $C_2 > \dfrac{2(-\alpha)(-\alpha-1)}{2(-\alpha)(-\alpha-1)+1}$;
- $\gamma_2 = \gamma_1$ for $C_2 = \dfrac{2(-\alpha)(-\alpha-1)}{2(-\alpha)(-\alpha-1)+1}$ and
- $\gamma_2 > \gamma_1$ for $C_2 < \dfrac{2(-\alpha)(-\alpha-1)}{2(-\alpha)(-\alpha-1)+1}$.

For $n \geq 2$ we have $\dfrac{2n(-\alpha-1)(1-\alpha-n)}{2n(-\alpha-1)(1-\alpha-n)+1} > \dfrac{2(-\alpha)(-\alpha-1)}{2(-\alpha)(-\alpha-1)+1}$. Then,

$$\gamma_{min} = \gamma_2 \ \text{ if } \ \frac{2(-\alpha)(-\alpha-1)}{2(-\alpha)(-\alpha-1)+1} < C_2 < \frac{2n(-\alpha-1)(1-\alpha-n)}{2n(-\alpha-1)(1-\alpha-n)+1}.$$

In order to compare γ_1 and γ_n observe that

$$\gamma_n - \gamma_1 = (n-1)(-\alpha-n) - C_2 \left[\frac{n(n-1)(1-\alpha-n)(-\alpha-n)-1}{n(1-\alpha-n)} + \frac{1}{-\alpha} \right].$$

Then $\gamma_n < \gamma_1$ if $C_2 > \dfrac{n(1-\alpha-n)(-\alpha)}{n(1-\alpha-n)(-\alpha)+1}$ and $\gamma_n \geq \gamma_1$ otherwise. On the other hand, for $n \geq 2$, we have

$$\frac{2(-\alpha)(-\alpha-1)}{2(-\alpha)(-\alpha-1)+1} < \frac{n(-\alpha)(1-\alpha-n)}{n(-\alpha)(1-\alpha-n)+1} < \frac{2n(-\alpha-1)(1-\alpha-n)}{2n(-\alpha-1)(1-\alpha-n)+1},$$

Thus,

- if $\dfrac{2n(-\alpha-1)(1-\alpha-n)}{2n(-\alpha-1)(1-\alpha-n)+1} < C_2 < 1$, then $\gamma_{min} = \gamma_n$;
- if $0 < C_2 < \dfrac{2(-\alpha)(-\alpha-1)}{n(-\alpha)(-\alpha-1)+1}$, then $\gamma_{min} = \gamma_1$.

We can again transform the restrictions for C_2 into equivalent restrictions for $D = C_1/C_2$. This procedure yields:

- if $[2(-\alpha)(-\alpha-1)]^{-1} < D < \infty$, then $\gamma_{min} = \gamma_1$;
- if $[2n(-\alpha-1)(1-\alpha-n)]^{-1} < D < [2(-\alpha)(-\alpha-1)]^{-1}$, then $\gamma_{min} = \gamma_2$;
- if $0 < D < [2n(-\alpha-1)(1-\alpha-n)]^{-1}$, then $\gamma_{min} = \gamma_n$.

Finally, we can remove the restriction $C_1 + C_2 = 1$ in all the cases $\nu = J, L, B$. This is justified just as in the final stage of the proof of Theorem 2 in [2]. ■

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