COMMENT

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Abstract. This is a very short comment on a conjecture on the zeros of ultraspherical polynomials posed by Árpád Elbert, Andrea Laforgia and Panayotis Siafarikas during the Fifth International Symposium on Orthogonal Polynomials, Special Functions and their Applications, held at the University of Patras, Greece, September 20-24, 1999.

1. The conjecture

In this section the conjecture is stated, as posed originally.

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A conjecture on the zeros of ultraspherical polynomials

For $k = 1, 2, \dots, [n/2]$ and $\lambda > -\frac{1}{2}$, let $x_{nk}^{(\lambda)}$ be the k-th positive zero, in decreasing order, of the ultraspherical polynomial $P_n^{(\lambda)}(x)$, of degree n, $n = 1, 2, \dots$. We formulate the following conjecture:

$$x_{nk}^{(\lambda)} \text{ is a convex function of } \lambda.$$

Remark. Elbert and Laforgia [1] proved that

$$\lim_{\lambda\to\infty} \lambda^{5/2} \frac{\partial^2}{\partial\lambda^2} x_{nk}^{(\lambda)} = \frac{3}{4} h_{n,k},$$

where $h_{n,k}$ is the k-th zero of the Hermite polynomial $H_n(x)$, of degree n.

Kokologiannaki and Siafarikas [2] proved the conjecture under the restriction $\lambda > \frac{n}{\sqrt{3}} + \frac{1}{2}$ and only for $k = 1$.

References

[1] Á. Elbert and A. Laforgia, *Asymptotic formulas for ultraspherical polynomials $P_n^{(\lambda)}(x)$ and their zeros, for large values of λ*, Proc. Amer. Math. Soc. **114** (1992), 371–377.

[2] C. G. Kokologiannaki and P. Siafarikas, *Convexity of the largest zero of the ultraspherical polynomials*, Integral Transform and Special Functions, vol. 4 (1) (1996), 1–6.

2. The Comment

Numerical experiments show that the above conjecture fails to hold for the largest zero $x_{n1}(\lambda) := x_{n1}^{(\lambda)}$ when λ is small and n is large enough. We provide two arguments in support of our statement.

The first one is as follows. Observe that $x_{n1}(-1/2) = 1$ for every natural n. Since the zeros of $P_n^{(\lambda)}(x)$ coincide with the zeros of the Chebyshev polynomials of the first and of the second kind for $\lambda = 0$ and for $\lambda = 1$, respectively, then

$x_{n1}(0) = \cos(\pi/2n)$ and $x_{n1}(1) = \cos(\pi/(n+1))$. If $x_{n1}(\lambda)$ is convex, then the expression

$$\mu x_{n1}(\lambda_1) + (1-\mu)x_{n1}(\lambda_2) - x_{n1}(\mu\lambda_1 + (1-\mu)\lambda_2)$$

must be positive for each $\mu \in [0,1]$ and for every pair of real parmeters $\lambda_1, \lambda_2 \geq -1/2$. For $\mu = 2/3$, $\lambda_1 = -1/2$ and $\lambda_2 = 1$ the above expression reduces to

$$(2/3) + (1/3)\cos(\pi/(n+1)) - \cos(\pi/2n),$$

which is positive only for $n = 1, \ldots, 6$, and negative for $n > 6$.

Various numerical experiments show that, when n is sufficiently large and fixed, the function $x_{n1}(\lambda)$ is concave in some interval $-1/2 < \lambda < \lambda_0(n)$ and convex only for $\lambda > \lambda_0(n)$. Execute the simple MATHEMATICA 3.0 program

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tab1 = Table[N[FindRoot[GegenbauerC[10, -0.5 + k * Sqrt[2]/50, x]
== 0, {x, 1}], 16], {k, 1, 100}];
Table[N[tab1[[k - 1, 1, 2]] + tab1[[k + 1, 1, 2]] - 2 * tab1[[k, 1, 2]], 16],
{k, 2, 99}]
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The first command determines approximately the largest zeros of $P_{10}^{(\lambda)}(x)$ by the Newton's method with an initial approximation $x_0 = 1$, when λ takes values at the arithmetic mesh $-0.5 + k\varepsilon,\ k = 1, \ldots, 100$, with $\varepsilon = \sqrt{2}/50$. The second command calculates the second finite differences of $x_{10,1}(\lambda)$ at the mesh points. The first 62 numbers in the resulting table are negative and the remaining one are positive. This shows that $x_{10,1}(\lambda)$ is concave for $-1/2 < \lambda < \lambda_0(10)$ and convex for $\lambda > \lambda_0(10)$, where $\lambda_0(10) \approx 1.267766$.

Kokologiannaki and Siafarikas' result provides the upper bound $n/\sqrt{3} + 1/2$ for $\lambda_0(n)$. However, the above arguments show that their theorem can not be extended to the whole range of λ. Some additional examples as well as positive results on convexity and concavity properties of $x_{nk}(\lambda)$ will appear elsewhere.

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