Counterexamples to a Problem of Pólya and to a Problem of Karlin *

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Abstract

Simple counterexamples to two open problems, due to Pólya and to Karlin, discussed very recently in [4], are given.

1 INTRODUCTION

Two open problems, due to Pólya and to Karlin, were discussed in detail in the recent paper [4]. Here we only recall some definitions and denotations of [4] and the statements of the problems in a form of conjectures.

The finite sequence of functions $f_0, \ldots, f_n$ obeys Descartes' rule in the interval (a, b) if the number of zeros in (a, b) of any real nonzero linear combination

$$\alpha_0 f_0(x) + \ldots + \alpha_n f_n(x)$$

does not exceed the number of sign changes in the sequence $\alpha_0, \ldots, \alpha_n$.
CONJECTURE 1 (Pólya) *For every polynomial p of precise degree n with distinct real zeros $\eta_1 < \ldots < \eta_n$ the sequence $p, p', \ldots, p^{(n)}$ obeys Descartes' rule in (η_n, ∞).*

A real entire function $\varphi(x)$,

$$\varphi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!},$$

is said to be in the *Laguerre-Pólya class* if $\varphi(x)$ can be represented in the form

$$\varphi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k},$$

where c, β, x_k are real, $\alpha \geq 0, m$ is a nonnegative integer and $\sum x_k^{-2} < \infty$. If φ is in the Laguerre-Pólya class, we write $\varphi \in \mathcal{L} - \mathcal{P}$. In particular, every real polynomial with only real zeros is in $\varphi \in \mathcal{L} - \mathcal{P}$. In this case we set $\alpha = \beta = 0$ and $x_k = \infty$ for all but a finite number of x_k.

For any $2k-2$ times continuously differentiable function f, the $k \times k$ Hankel determinant $H_k(f; x)$ is defined by

$$H_k(f; x) = det(f^{(i+j)}(x))_{i,j=0}^{k-1}.$$

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CONJECTURE 2 (Karlin [5]) *For every $\varphi \in \mathcal{L}-\mathcal{P}$ with nonnegative Maclaurin coefficients γ_k and for any $q = 0, 1, \ldots$ and $k = 2, 3, \ldots$,*

$$(-1)^{k(k-1)/2} H_k(\varphi^{(q)}; x) \geq 0 \quad \textit{for} \quad x \geq 0.$$

We recall also that Karlin's problem, as formulated here, appears as *Open Problem* 3.4 in [2].

A "proof" that the statement of Pólya's conjecture, if true, implies the truth of Karlin's one was given in [4]. Meanwhile, it was pointed out to the author by George Csordas that that "proof" does not work as given, because of a use of an incorrect result of Csordas and Williamson [3] concerning simplicity of the zeros of Jensen polynomials associated with an entire function in the Laguerre-Pólya class.

A relatively simple counterexample to the above conjectures is given in Section 3. In order to do this, we give an equivalent form of Pólya's conjecture in Section 2. It is shown also that our proof of Theorem 1 in [4] actually works with a slight modification allowing us to avoid the use of the result of [3]. In Section 4 we indicate a subclass of polynomials with real nonpositive zeros, for which Pólya's conjecture holds. Similarly, a subclass of $\mathcal{L}-\mathcal{P}$ is extracted, such that for every function in the subclass, Karlin's conjecture holds.

While the manuscript was under consideration the author was kindly informed by Professors Tomas Craven and George Csordas that they found counterexamples to Pólya's and to Karlin's conjectures. They constructed also a counterexample to the latter conjecture in the case when the entire function is a transcendental one. Their methods of proofs are different from the approach used in the present paper.

2 An equivalent form of Pólya's conjecture

Recall that a matrix is said to be totally positive if all its minors are nonnegative. The basic tool in the proofs of our results is the following fundamental characterisation of sequences of polynomials, which obey Descartes' rule of signs, due to Schoenberg:

THEOREM 1 (Schoenberg [8]) *The sequence of polynomials*

$$\begin{aligned}
p_0(x) &= a_{00}, \\
p_1(x) &= a_{10} + a_{11}x, \\
p_2(x) &= a_{20} + a_{21}x + a_{22}x^2, \\
\vdots \quad & \quad \ddots \\
p_n(x) &= a_{n0} + a_{n1}x + \cdots + a_{nn}x^n,
\end{aligned}$$

with $a_{00} > 0, a_{11} > 0, \ldots, a_{nn} > 0$, obeys Descartes' rule of signs in $(0, \infty)$ if and only if the upper triangular matrix

$$\begin{pmatrix} a_{00} & a_{10} & \cdots & a_{n0} \\ & a_{11} & \cdots & a_{n1} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$$

is totally positive.

An almost immediate consequence of Theorem 1 are the following equivalent forms of Pólya's conjecture:

Lemma 1 *The following statements are equivalent:*

(i) For every polynomial p of precise degree n with distinct real zeros $\eta_1 < \ldots < \eta_n$, the sequence $p, p', \ldots, p^{(n)}$ obeys Descartes' rule in (η_n, ∞).

(ii) For every polynomial p,

$$\begin{aligned} p(x) &= x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ &= (x-\eta_1)\cdots(x-\eta_n), \end{aligned}$$

whose zeros $\eta_1, \ldots, \eta_n$ are nonpositive, the sequence $p, p', \ldots, p^{(n)}$ obeys Descartes' rule in $(0, \infty)$.

(iii) For every polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with real nonpositive zeros, the upper triangular matrix

$$\begin{pmatrix} n! & (n-1)!a_{n-1} & \cdots & 1!a_1 & a_0 \\ & n! & \ddots & \ddots & 1!a_1 \\ & & \ddots & \ddots & \vdots \\ & & & n! & (n-1)!a_{n-1} \\ & & & & n! \end{pmatrix} \tag{1}$$

is totally positive.

Observe that (i) is simply the statement of Pólya's conjecture and that neither in (ii) nor in (iii) we require the zeros of $p(x)$ to be distinct.

Proof. It is clear that there is no loss of generality if we normalise the polynomial p to be monic and suppose that $\eta_n \le 0$ in statement (i).

Note that the matrix (1) is totally positive if and only if the $(n+1)\times(n+1)$ matrix

$$\begin{pmatrix} n! & (n-1)!a_{n-1} & \cdots & 2a_2 & a_1 & a_0 \\ & n! & \ddots & 6a_3 & 2a_2 & a_1 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & n(n-1) & (n-1)a_{n-1} & a_{n-2} \\ & & & & n & a_{n-1} \\ & & & & & 1 \end{pmatrix} \tag{2}$$

is totally positive. Indeed, the matrix (1) is obtained from (2) by multiplying the k-th row by $(k-1)!$, for $k = 1, \ldots, n+1$.

Thus Theorem 1 gives the equivalence of (ii) and (iii). Theorem 1 yields also the implication (iii)⇒(i).

Suppose (i) holds for any

$$\begin{aligned} p(x) &= x^n + a_{n-1}x^{n-1} + \cdots + a_1 x \\ &= (x-\eta_1)\cdots(x-\eta_n), \end{aligned}$$

with $\eta_1 < \ldots < \eta_{n-1} < \eta_n \le 0$. Then, by Theorem 1, every minor of (2), or equivalently every minor Δ of (1), is nonnegative. Obviously Δ is a continuous function of $\eta_1, \ldots, \eta_n$. Since it is nonnegative in the open simplex $\Xi = \{\eta_1 < \ldots < \eta_n < 0\}$, then it is nonnegative on the closure of Ξ and (iii) holds. □

Despite that we give counterexamples to the conjectures, we find it necessary to fix the proof of Theorem 1 in [4]. It is not difficult to see that with the aid of Lemma 1 we can avoid the use of the result in [3]. Indeed, a closer inspection shows that we needed the simplicity of the zeros of Jensen polynomials $g_n(z)$, associated with an entire function in $\mathcal{L}-\mathcal{P}$ with positive Maclaurin coefficients, in order to conclude, via the statement of Conjecture 1, that the sequence $g_n, g_n', \ldots, g_n^{(n)}$ (or equivalently $p_n, p_n', \ldots, p_n^{(n)}$, where $p_n(z) = g_n(z/n)$) obeys Descartes' rule in $(0,\infty)$. Observe that, because of the equivalence of (i) and (ii) in Lemma 1, the above sequences obey Descartes rule in $(0,\infty)$ subject to the validity of Conjecture 1, no matter η_k are distinct or not.

3 The counterexamples

First we prove:

THEOREM 2 *The sequence $p, p', \ldots, p^{(8)}$, where $p(x) = x^6(x+1)^2$ does not obey Descartes' rule in $(0,\infty)$.*

Proof. In view of the equivalence of (ii) and (iii) in Lemma 1, it suffices to show that, for $n = 8$, at least one minor of the upper triangular, tridiagonal $(n+1)\times(n+1)$ matrix

$$\begin{pmatrix} n! & 2(n-1)! & (n-2)! & & & \\ & n! & 2(n-1)! & (n-2)! & & \\ & & \ddots & \ddots & \ddots & \\ & & & n! & 2(n-1)! & n-2)! \\ & & & & n! & 2(n-1)! \\ & & & & & n! \end{pmatrix},$$

is negative. We show that the $n \times n$ minor

$$det \begin{pmatrix} 2(n-1)! & (n-2)! & & & \\ n! & 2(n-1)! & (n-2)! & & \\ & \ddots & \ddots & \ddots & \\ & & n! & 2(n-1)! & (n-2)! \\ & & & n! & 2(n-1)! \end{pmatrix} \tag{3}$$

is negative. The sign of this minor is the same as the sign of the determinant $\Delta_{8,8}$, where $\Delta_{n,k}$ is the $k \times k$ determinant

$$\Delta_{n,k} = det \begin{pmatrix} 2(n-1) & 1 & & & \\ n(n-1) & 2(n-1) & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & n(n-1) & 2(n-1) & 1 \\ & & & n(n-1) & 2(n-1) \end{pmatrix}.$$

Obviously $\Delta_{n,k}$ satisfy the recurrence relation

$$\begin{aligned} \Delta_{n,0} &= 1, \\ \Delta_{n,1} &= 2(n-1), \\ \Delta_{n,k+1} &= 2(n-1)\Delta_{n,k} - n(n-1)\Delta_{n,k-1}. \end{aligned} \tag{4}$$

Observe that it is a recurrence relation with respect to the indices k with constant coefficients. Then the general solution of (4) has the form $\Delta_{n,k} = c_1 z_1^k + c_2 z_2^k$, where $z_{1,2} = (n-1) \pm i\sqrt{n-1}$ are the solutions of the characteristic equation $z^2 - 2(n-1)z + n(n-1) = 0$ of (4). The constants c_1 and c_2 and uniquely determined by the initial conditions for $\Delta_{n,0}$ and $\Delta_{n,1}$. We have $c_{1,2} = (1 \mp i\sqrt{n-1})/2$. Now straightforward calculations yield

$$\Delta_{n,k} = \frac{(n-1)^{k/2}}{2i} \left\{ (\sqrt{n-1} + i)^{k+1} - (\sqrt{n-1} - i)^{k+1} \right\}.$$

In particular,

$$\Delta_{8,8} = -\frac{7^4}{2} 2560,$$

which proves the theorem.

By the continuity argument from the proof of Lemma 1 we can conclude that:

Corollary 1 *There exist a class of polynomials $p(x)$ of degree 8 with distinct real zeros $\eta_1 < \eta_2 < \ldots < \eta_7 < \eta_8 = 0$, such that the zeros η_1 and η_2 are sufficiently close to -1, the zeros $\eta_3 < \eta_4 < \ldots < \eta_7$ are very close to the origin and $p, p', \ldots, p^{(8)}$ does not obey Descartes' rule in $(0, \infty)$.*

This is exactly the counterexample to the problem of Pólya.

We omit the computational details. It is worth mentioning, however, that the general formula for $\Delta_{n,k}$ allows to construct a variety of counterexamples of this form. It seems that $\Delta_{n,n} < 0$ for every $n \geq 8$ and $\Delta_{n,k} < 0$ even for some $k < n$ if n is large. For example, for $n = 10$ the determinants $\Delta_{10,9}$ and $\Delta_{10,10}$ are negative.

The above counterexample to the problem of Pólya serves also as a counterexample to the problem of Karlin:

Corollary 2 *For $p(x) = x^6(x+1)^2$ and $k = 8$ we have*

$$(-1)^{k(k-1)/2} H_k(p; 0) < 0.$$

We need only to reorder the columns of $H_8(p;0)$ and to see that $(-1)^{28} H_8(p;0)$ coincides with the determinant (3), associated with $p(x)$ and thus the sign of $(-1)^{28} H_8(p;0)$ coincides with the sign of $\Delta_{8,8}$.

4 Polynomials and entire functions for which the conjectures hold

Despite the counterexamples to the conjectures, as raised by Pólya and Karlin, it is of interest to indicate less general classes of polynomials and entire functions in $\mathcal{L} - \mathcal{P}$, for which the conjectures hold. It is done in the present section.

Theorem 3 *Suppose that the polynomial*

$$q(x) = a_0 + 1!a_1 x + 2!a_2 x^2 + \cdots + n!a_n x^n$$

has nonnegative coefficients and only real nonpositive zeros. Then the polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has also only real nonpositive zeros. Moreover, the sequence $p, p', \ldots, p^{(n)}$ obeys Descartes rule in $(0, \infty)$.

We need the following definition:

Definiton 1 *A real sequence* $\{\alpha_k\}_{k=0}^{\infty}$ *is said to be totally positive sequence if* $\sum_{k=0}^{\infty}\alpha_k x^k$ *is an entire function and the infinite lower triangular matrix*

$$\begin{pmatrix} \alpha_0 & & & & \\ \alpha_1 & \alpha_0 & & & \\ \alpha_2 & \alpha_1 & \alpha_0 & & \\ \ddots & \ddots & \ddots & \ddots & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

is totally positive.

A key role in the proofs of the results in this section is played by the following characterisation of functions in the Laguerre-Pólya class with nonnegative Maclaurin coefficients in terms of totally positive sequences, due to Aisen, Edrei, Schoenberg and Whitney [1]:

Theorem 4 *The real entire function* $\psi(x)=\sum_{k=0}^{\infty}\alpha_k x^k$ *with nonnegative coefficients* α_k *is in the Laguerre-Pólya class if and only if* $\{\alpha_k\}_{k=0}^{\infty}$ *is totally positive.*

As an immediate consequence we obtain:

Corollary 3 *The real polynomial* $r(x)=\sum_{k=0}^{n}\alpha_k x^k$ *with nonnegative coefficients* α_k *has only real nonpositive zeros if and only if the infinite lower triangular matrix*

$$\begin{pmatrix} \alpha_0 & & & & & & \\ \alpha_1 & \alpha_0 & & & & & \\ \alpha_2 & \alpha_1 & \alpha_0 & & & & \\ \vdots & \ddots & \ddots & \ddots & & & \\ \alpha_n & \ddots & \ddots & \ddots & \alpha_0 & & \\ & \alpha_n & \ddots & \ddots & \ddots & \alpha_0 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

is totally positive.

It is worth mentioning that the totally positive sequences are in particular Pólya frequency sequences. We refer to Chapter 8 of [5] for the definition and more information about Pólya frequency sequences as well as for results related to Theorem 4 and Corollary 3.

Proof of Theorem 3. The first assertion of the theorem is nothing but the statement of Malo's theorem (cf. Marden [6] and Obrechkoff [7]). Since $q(x)$ has nonnegative coefficients and only real nonpositive zeros, then by Corollary 3 the sequence $a_0, 1!a_1, 2!a_2, \ldots, n!a_n$ is totally positive. In particular the finite matrix

$$\begin{pmatrix} n! & (n-1)!a_{n-1} & \cdots & 1!a_1 & a_0 \\ & n! & \ddots & \ddots & 1!a_1 \\ & & \ddots & \ddots & \vdots \\ & & & n! & (n-1)!a_{n-1} \\ & & & & n! \end{pmatrix}$$

is totally positive. Now we can infer by Lemma 1 that the sequence $p, p', \ldots, p^{(n)}$ obeys Descartes' rule of signs in $(0,\infty)$.

THEOREM 5 *Let the real entire function*

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k x^k$$

with nonnegative coefficients γ_k *be in* $\mathcal{L}-\mathcal{P}$. *Then*

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$$

is also in $\mathcal{L}-\mathcal{P}$. *Moreover, for any* $q = 0, 1, \ldots,$ *and* $k = 2, 3, \ldots$

$$(-1)^{k(k-1)/2} H_k(\varphi^{(q)}; x) \geq 0 \quad \text{for} \quad x \geq 0.$$

First we provide only a sketch of a proof. The first assertion is a straightforward consequence of Theorem 3. Indeed, since the Jensen polynomials associated with $\psi^{(q)}$ have only real nonpositive zeros, then by Theorem 3, all the zeros of the Jensen polynomials $g_{n,q}$ associated with $\varphi^{(q)}$ are also nonpositive. Furthermore, Theorem 3 implies also that sequence of polynomials, formed by $g_{n,q}$ and its derivatives, obeys Descartes' rule in $(0, \infty)$. It remains only to note that proving Theorem 1 in [4] (the proof is fixed in Section 2 of the present paper) we proved actually the following: If for any positive integer n the sequence of polynomials, formed by the Jensen polynomial g_n of degree n associated with the entire function in $\mathcal{L}-\mathcal{P}$ and the consecutive derivatives $g_n', \ldots, g_n^{(n)}$ of g_n obeys Descartes rule in $(0, \infty)$, then, for $k = 2, 3, \ldots$

$$(-1)^{k(k-1)/2} H_k(\varphi; x) \geq 0 \quad \textit{for} \quad x \geq 0.$$

Another, formally different proof of Theorem 5 can be given by the use of Theorem 4 only. Since ψ is an entire function in $\mathcal{L}-\mathcal{P}$ with nonnegative coefficients, then Theorem 4 implies that the infinite lower triangular matrix

$$A = \begin{pmatrix} \gamma_0 & & & & \\ \gamma_1 & \gamma_0 & & & \\ \gamma_2 & \gamma_1 & \gamma_0 & & \\ \ddots & \ddots & \ddots & \ddots & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

is totally positive.

In order to prove Theorem 5 we show that the Maclaurin coefficients of the entire function

$$\begin{aligned} H(x) &= (-1)^{k(k-1)/2} H_k(\varphi^{(q)}; x) \\ &= det \begin{pmatrix} \varphi^{(q+k-1)}(x) & \varphi^{(q+k-2)}(x) & \cdots & \varphi^{(q+1)}(x) & \varphi^{(q)}(x) \\ \varphi^{(q+k)}(x) & \varphi^{(q+k-1)}(x) & \cdots & \varphi^{(q+2)}(x) & \varphi^{(q+1)}(x) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \varphi^{(q+2k-2)}(x) & \varphi^{(q+2k-3)}(x) & \cdots & \varphi^{(q+k)}(x) & \varphi^{(q+k-1)}(x) \end{pmatrix} \end{aligned}$$

are nonnegative. Note that

$$H(0) = det \begin{pmatrix} \gamma_{q+k-1} & \gamma_{q+k-2} & \cdots & \gamma_{q+1} & \gamma_q \\ \gamma_{q+k} & \gamma_{q+k-1} & \cdots & \gamma_{q+2} & \gamma_{q+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \gamma_{q+2k-2} & \gamma_{q+2k-3} & \cdots & \gamma_{q+k} & \gamma_{q+k-1} \end{pmatrix}.$$

The latter determinant is a minor of A and then $H(0) \geq 0$.

Differentiating the determinant which represents $H(x)$ row by row, we see that the derivative of H is represented by a determinant which differs from that of H only by its last row, where the order of the derivatives is shifted by one. Hence $H'(0)$ is also a determinant which is a minor of A and $H'(0) \geq 0$. The second derivative of H is a sum of two determinants. The first determinant differs from the one of H only by its last row where the orders of the derivatives are shifted by two each. The second determinant differs from $H(x)$ by the last two rows and in each of these rows the orders of the derivatives are shifted by one each. Then $H''(0)$ is a sum of two determinants which are again minors of A and this yields $H''(0) \geq 0$.

It can be seen that for any positive integer m the derivative $H^{(m)}(x)$ is a sum of determinants and each of them consists of rows whose elements are derivatives of $\varphi(x)$ whose orders are consecutive in decreasing order. The orders of the derivatives of $\varphi(x)$ in the first, and thus in each, column are not necessarily consecutive but always form an increasing sequence. Hence $H^{(m)}(0)$ is a sum of determinants, each of which is a minor of A. Then $H^{(m)}(0) \geq 0$. Therefore $H(x) \geq 0$ for $x \geq 0$.

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