A Problem of Pólya Concerning Polynomials Which Obey Descartes' Rule of Signs *

Dimitar K. Dimitrov †

Departamento de Ciências de Computação e Estatística, IBILCE, Universidade Estadual Paulista, 15054-000 São José do Rio Preto, SP, Brazil

Abstract

We formulate and duscuss two open problems. The first one is due to Pólya. It states that the sequence of polynomials formed by a polynomial p with only real zeros and all its derivatives, obeys Descartes' rule of signs for any x, greater than the largest zero of p. The other problem is due to Karlin and states that certain Hankel determinants associated with an entire function in the Laguerre-Polya class do not change their signs. We prove that the statement of Karlin's problem is a consequence of that of Pólya's problem. The interest in these problems is motivated by the fact that once Karlin's problem is solved, it would yield necessary conditions that the Riemann hypothesis holds.

1 Introduction and statement of the problems

We consider an old open problem of Pólya, which states that for any polynomial p of degree n with distinct real zeros $\eta_1 < \ldots < \eta_n$ the sequence $p, p', \ldots, p^{(n)}$ obeys Descartes' rule in (η_n, ∞). We prove that another open problem, due to Karlin, can be immediately solved after solving Pólya's problem. Karlin's problem states that certain Hankel determinants associated with an entire function in the Laguerre-Pólya class alternate in sign. The connection of Karlin's problem, and thus of Pólya's one, to the Riemann hypothesis is also discussed.

In order to formulate the problems we need some definitions. The finite sequence of functions $f_0, \ldots, f_n$ obeys Descartes' rule in the interval (a, b) if the number of zeros in (a, b) of any real nonzero linear combination

$$\alpha_0 f_0(x) + \ldots + \alpha_n f_n(x)$$

does not exceed the number of sign changes in the sequence $\alpha_0, \ldots, \alpha_n$.

Problem 1 (Pólya) *Let p be a polynomial of precise degree n with distinct real zeros $\eta_1 < \ldots < \eta_n$. Prove that the sequence $p, p', \ldots, p^{(n)}$ obeys Descartes' rule in (η_n, ∞).*

This problem appears in a footnote of a celebrated paper of Schoenberg [19]. Schoenberg atributes it to Pólya. The statement of the problem is very natural. Note that the functions $1, x, \ldots, x^n$ obey Descartes' rule in $(0, \infty)$ and the proof of this fact goes back to Descartes. A well-known result of

*Research supported by the Brazilian foudation CNPq under Grant 300645/95-3 and the Bulgarian Science Foundation under Grant MM-414.

†On leave from The University of Rousse, Bulgaria

Runge states that the sequence $1, x - x_1, (x - x_1)(x - x_2), \ldots, (x - x_1)(x - x_2)\cdots(x - x_n)$, where $x_1, x_2, \cdots x_n$ are arbitrary real numbers, obeys Descartes' rule for $x > max\{x_i : 1 \le i \le n\}$. The sequence of orthogonal polynomials $p_0, p_1, \ldots, p_n$, normalized in such a way that the leading coefficients of all the polynomials have the same sign, also obeys Descartes' rule for $x > \zeta_n$, where ζ_n is the largest zero of $p_n(x)$. The latter result, as stated here, was proved first by Obrechkoff [11, 12, 13] and rediscovered later by Schoenberg [19]. It is worth mentioning that the proofs of Obrechkoff and of Schoenberg are different. The fact that the sequence of Legendre polynomials $P_0, P_1, \ldots, P_n$ obeys the Descartes' rule for $x > \zeta_n$, where ζ_n is the largest zero of $P_n(x)$ was proved much earlier by Laguerre [9], and M. Marden [10] gave a proof of Obrechkoff-Schoenberg's result for the classical orthogonal polynomials. Laguerre's and Marden's proofs use the second order differential equation satisfied by the classical orthogonal polynomials. A well-known result of Bochner[2] (see also [3, p. 150]) states that the only orthogonal polynomials, associated with a real measure, which satisfy a second order differential equation are the classical one, so that Laguerre's and Marden's methods can not be extended to prove Obrechkoff-Schoenberg's result.

Observe that the latter three sequences of polynomials have common properties. The leading coefficients of all the polynomials have the same sign, all their zeros are real and the zeros of two consecutive polynomials interlace in the sence that, if $\xi_{k1} < \ldots < \xi_{kk}$ and $\xi_{k-1,1} < \ldots < \xi_{k-1,k-1}$ are the zeros of the k-th and (k-1)-th polynomials, then $\xi_{ki} \le \xi_{k-1,i} \le \xi_{k,i+1}$ for $i = 1, \ldots, k-1$. One might expect that Descartes', Runge's and Obrechkoff-Schoenberg's results can be extended to any sequence of polynomials with these properties. This question was posed by Pólya and appears in the same footnote of Schoenberg's paper [19] but Schoenberg gives a simple counterexample there. However, there still might exist other sequences of polynomials with interlacing zeros which obey Descartes' rule. One natural candidate is the sequence which appears in Problem 1.

A real entire function $\psi(x)$ is said to be in the *Laguerre-Pólya class* if $\psi(x)$ can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k}, \tag{1}$$

where c, β, x_k are real, $\alpha \ge 0, m$ is a nonnegative integer and $\sum x_k^{-2} < \infty$. If ψ is in the Laguerre-Pólya class, we write $\psi \in \mathcal{L} - \mathcal{P}$. A real entire function $\varphi(x)$ is a function of *type I in the Laguerre-Pólya class* if $\varphi(x)$ or $\varphi(-x)$ can be represented in the form

$$\varphi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\infty} (1 + x/x_k), \tag{2}$$

where c is real, $\sigma \ge 0, m$ is a nonnegative integer, $x_k > 0$, and $\sum 1/x_k < \infty$. If φ is of type I in the Laguerre-Pólya class, we shall write $\varphi \in \mathcal{L} - \mathcal{P}(I)$. We have adopted the notations of [4, 14, 18] where one may consult for the properties of functions in the Laguerre-Polya class. Generally, $\mathcal{L} - \mathcal{P}$ consists of entire functions which are uniform limits on the compact sets of the complex plane of polynomials with only real zeros, and the functions in $\mathcal{L} - \mathcal{P}(I)$ are uniform limits on the compact sets of the complex plane of polynomials whose zeros are real and are all positive, or all negative. It is clear that $\mathcal{L} - \mathcal{P}(I) \subset \mathcal{L} - \mathcal{P}$. An important observation of Pólya and Schur [17] is that, if a function

$$\varphi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{3}$$

is in $\mathcal{L} - \mathcal{P}$ and its Maclaurin coefficients $\gamma_k, \quad k = 0, 1, \ldots$, are nonnegative, then $\varphi \in \mathcal{L} - \mathcal{P}(I)$, and, moreover, the constant term c in the representation of φ in the form (2) is positive.

For any $2k-2$ times continuously differentiable function f, the $k \times k$ Hankel determinant $H_k(f;x)$ is defined by

$$H_k(f;x) = det(f^{(i+j)}(x))_{i,j=0}^{k-1}.$$

PROBLEM 2 (Karlin [8]) *Let $\varphi \in \mathcal{L}-\mathcal{P}$ and its Maclaurin coefficients γ_k be nonnegative. Prove that for any $q = 0, 1, \ldots$ and $k = 2, 3, \ldots$*

$$(-1)^{k(k-1)/2} H_k(\varphi^{(q)};x) \geq 0 \quad \textit{for} \quad x \geq 0. \tag{4}$$

As it was pointed out by Craven and Csordas [4, p. 258], the conjecture appears with a misprint on pp. 389-190 in [8]. Karlin's conjecture, as stated above was formulated as *Open Problem* 3.4 in [4].

Note that for $k=2$ inequalities (4) reduce to the well-known Laguerre inequalities

$$(\varphi^{(q)}(x))^2 - \varphi^{(q-1)}(x)\varphi^{(q+1)}(x) \geq 0, \tag{5}$$

which hold even for all real x and for entire functions in $\mathcal{L}-\mathcal{P}$. Craven and Csordas [4] proved inequalities (4) for $k=3$.

Particularly interesting case of inequalities (4) is when $x=0$. In this case (4) reads as

$$(-1)^{k(k-1)/2} det \begin{pmatrix} \gamma_q & \gamma_{q+1} & \cdots & \gamma_{q+k-1} \\ \gamma_{q+1} & \gamma_{q+2} & \cdots & \gamma_{q+k} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{q+k-1} & \gamma_{q+k} & \cdots & \gamma_{q+2k-2} \end{pmatrix} \geq 0. \tag{6}$$

Note that while the conjectured inequalities (4) can be considered as natural extensions of Laguerre's inequalities, inequalities (6) are natural extensions of the so-called Turán inequalities

$$\gamma_q^2 - \gamma_{q-1}\gamma_{q+1} \geq 0, \qquad q = 1, 2, \ldots, \tag{7}$$

which are necessary conditions that an entire function of the form (3) is in the Laguerre-Pólya class. The latter fact follows immediately from Laguerre's inequalities (5) for $x=0$.

2 Connection of the problems to the Riemann hypothesis

One of the important reasons for the interest in Karlin's problem is its close relation to the celebrated Riemann hypothesis concerning the zeros of the Riemann ζ-function. The Riemann $\xi-$function is defined by

$$\xi(iz) = \frac{1}{2}(z^2 - 1/4)\pi^{-z/2-1/4}\Gamma(z/2+1/4)\zeta(z+1/2),$$

where ζ is the Riemann zeta-function and Γ is the gamma-function. It is well-known that ξ is an entire function of order one. It can be represented in the from

$$\xi(x/2) = 8\int_0^\infty \Phi(t)\cos xt dt,$$

where

$$\Phi(t) = \sum_{n=1}^{\infty}(2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t})\exp(-n^2\pi e^{4t}). \tag{8}$$

Then

$$\frac{1}{8}\xi(x/2) = \sum_{k=0}^{\infty}(-1)^k \hat{b}_k \frac{x^{2k}}{(2k)!} \tag{9}$$

with

$$\hat{b}_k = \int_0^{\infty} t^{2k}\Phi(t)dt, \quad k = 0, 1, \ldots. \tag{10}$$

On setting $z = -x^2$ we obtain the entire function

$$F(z) = \sum_{k=0}^{\infty} \hat{b}_k \frac{z^k}{(2k)!} = \sum_{k=0}^{\infty} \hat{\gamma}_k \frac{z^k}{k!}, \quad \hat{\gamma}_k = \frac{k!}{(2k)!}\hat{b}_k. \tag{11}$$

of order $1/2$. Therefore, by the Hadamard theorem, the Riemann hypothesis is equivalent to the statement that the real entire function F, defined by (8), (10) and (11), is in $\mathcal{L}-\mathcal{P}(I)$ (cf. Polya and Schur [17] and Boas [1, p. 24]). Thus, once Karlin's conjecture is proved, it would provide necessary conditions that the Riemann hypothesis is true. These conditions would be of the form

$$(-1)^{k(k-1)/2} H_k(F^{(q)}; x) \geq 0 \quad \textit{for} \quad x \geq 0 \quad \textit{for any} \quad q = 0, 1, \ldots \quad \textit{and} \quad k = 2, 3, \ldots, \tag{12}$$

and, in particular, for $x = 0$

$$(-1)^{k(k-1)/2} det \begin{pmatrix} \hat{\gamma}_q & \hat{\gamma}_{q+1} & \cdots & \hat{\gamma}_{q+k-1} \\ \hat{\gamma}_{q+1} & \hat{\gamma}_{q+2} & \cdots & \hat{\gamma}_{q+k} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\gamma}_{q+k-1} & \hat{\gamma}_{q+k} & \cdots & \hat{\gamma}_{q+2k-2} \end{pmatrix} \geq 0 \quad \textit{for any} \quad q = 0, 1, \ldots \quad \textit{and} \quad k = 2, 3, \ldots. \tag{13}$$

In 1927 Pólya [16] conjectured that Turán inequalities hold for the Maclaurin coefficients of F. Precisely, because of (11), Pólya's conjecture reads as

$$(2q+1)(\hat{b}_q)^2 - (2q-1)\hat{b}_{q-1}\hat{b}_{q+1} > 0, \quad q = 1, 2, \ldots.$$

Almost sixty years later Csordas, Norfolk and Varga [5] proved the latter inequalities, thus verifying Pólya's conjecture. We refer the reader to a nice account of Varga [20, Chapter 3] about the proof of the conjecture and some other theoretical and computational aspects of the Riemann hypothesis.

It has already been mentioned that Craven and Csordas [4] proved Karlin's conjecture for $k = 3$. They pointed out also [4, Eq. (2.63)] that inequalities (6) hold true for $k = 3$ for any entire function from $\mathcal{L}-\mathcal{P}$ with nonnegative Maclaurin coefficients $\gamma_q, \quad q = 0, 1, \ldots$.

Thus, a natural extension of the above mentioned Pólya's conjecture is the following one:

Conjecture 1 *If $\hat{\gamma}_q, \quad q = 0, 1, \ldots$ are defined by (8), (10) and (11), then the inequalities*

$$det \begin{pmatrix} \hat{\gamma}_q & \hat{\gamma}_{q+1} & \hat{\gamma}_{q+2} \\ \hat{\gamma}_{q+1} & \hat{\gamma}_{q+2} & \hat{\gamma}_{q+3} \\ \hat{\gamma}_{q+2} & \hat{\gamma}_{q+3} & \hat{\gamma}_{q+4} \end{pmatrix} \leq 0 \tag{14}$$

hold for any nonnegative integer q.

Numerical experiments based on the first twenty coefficients $\hat{b}_q$ given by Csordas, Norfolk and Varga [5] suggest that the conjecture is true.

3 Pólya's problem implies Karlin's problem

In this section we prove that if the statement of Problem 1 is true, then the statement of Problem 2 is also true. Formally we state:

THEOREM 1 *Suppose that the statement of Problem 1 is true. Let φ be in $\mathcal{L}-\mathcal{P}$ and, let its Maclaurin coefficients γ_k be nonnegative. Then inequality (4) holds for any $q=0,1,\ldots$ and $k=2,3,\ldots$.*

If $f_0,\ldots,f_n$ are sufficiently smooth functions the Wronskian $W(f_0,\ldots,f_n;x)$ of $f_0,\ldots,f_n$ at x is defined by

$$W(f_0,\ldots,f_n;x)=det\left(f_j^{(i)}(x)\right)_{i,j=0}^{n}.$$

A basic tool in the proof is the following important result of Pólya and Szegő:

THEOREM 2 (Pólya and Szegö [18]) *The necessary and sufficient condition that the sequence of n-times continuously differentiable functions $f_0,\ldots,f_n$, none of which is identically zero in (a,b) obeys Descartes' rule in (a,b) is that for any integers $\nu_1,\ldots,\nu_k$ with $0\le\nu_1<\ldots<\nu_k\le n$ the Wronskian $W(f_{\nu_1},\ldots,f_{\nu_k})$ does not vanish in (a,b) and any two Wronskians with the same number k of columns have the same sign, for $k=1,\ldots,n$.*

It might be worth pointing out that the requirement that none of the functions is identically zero in (a,b) is essential. In [18] it was tacitly accepted.

Proof of Theorem 1. Since the function φ is in the Laguerre-Polya class and its Maclaurin coefficients are nonnegative, then there exists a sequence of polynomials $\{p_n\}$ with only nonpositive zeros such that $\{p_n\}$ converges locally uniformly (uniformly on every compact set of the complex plane) to φ. Craven and Csordas [4, Lemma 2.2] proved that the sequence $\{g_n(z/n)\}$, where

$$g_n(z):=\sum_{k=0}^{n}\binom{n}{k}\gamma_k z^k$$

are the Jensen polynomials, associated with φ, converges locally uniformly to φ. On the other hand, if φ is in $\mathcal{L}-\mathcal{P}$, then, for any positive integer n, the polynomial $g_n(z)$ has only real zeros [17] (see also [4] and [6]). Since the coefficients of $g_n(z)$ are nonnegative, then all its zeros are nonpositive. Thus, we may chose

$$p_n(z)=g_n(z/n). \tag{15}$$

Since the convergence of $p_n(z)$ to $\varphi(z)$ is uniform on the compact subsets of the complex plane, then, for any positive integer j, the sequence $\{p_n^{(j)}(z)\}_{n=1}^{\infty}$ converges locally uniformly to $\varphi^{(j)}(z)$. By Rolle's theorem the polynomials $p_n^{(j)}(z)$ have only real nonpositive zeros. Therefore $\varphi^{(j)}(z)$ also belongs to $\mathcal{L}-\mathcal{P}$. Thus, we have proved that $\mathcal{L}-\mathcal{P}$ is closed under differentiation. The first proof of this fact goes back to Pólya and Schur [17].

Note that the Jensen polynomials, associated with $\varphi^{(j)}(z)$ are

$$g_{n,j}(z):=\sum_{k=0}^{n}\binom{n}{k}\gamma_{k+j}z^k.$$

Thus, if $\varphi\in\mathcal{L}-\mathcal{P}$ and its Maclaurin coeficients are nonnegative, then all the polynomials $g_{n,j}(z),\ n,j=0,1,2,\ldots,$ have only real and nonnegative zeros.

Having in mind these observations, we can prove that the sequence $\gamma=\{\gamma_k\}$ of the Maclaurin coefficients $\gamma_k,\ \gamma_k\ge 0$ of a function in $\mathcal{L}-\mathcal{P}$ can not have "gaps". This means that γ can not have a subsequence

$$\gamma_j\neq 0,\ \gamma_{j+1}=\cdots=\gamma_{j+n-1}=0,\ \gamma_{j+n}\neq 0,\ n\ge 2.$$

Indeed, if such a subsequence existed, since γ_j and γ_{j+n} are necessarily positive, then $g_{n,j}(z)$ would have had nonreal zeros.

Thus, if $\varphi \in \mathcal{L}-\mathcal{P}$, its Maclaurin coefficients are nonnegative, φ is not a polynomial itself, and if $n \geq m$, where m is the power of x in (2), then the associated Jensen polynomial g_n has positive leading coefficient and exactly n nonnegative zeros. In what follows, to the end of the proof, we suppose that φ is not a polynomial and $n \geq m$.

We consider two cases. The first one is when $m = 0$ in the representation (2) of φ, and φ is not of the form $\varphi = cx^m e^{\sigma x}$. Csordas and Williamson [7] proved that in this case the zeros of the corresponding Jensen polynomials are simple. Thus, the polynomial p_n, defined by (15) has also simple negative zeros and positive leading coefficient. Let $n \geq q+k-1$. By the assumption that the statement of Problem 1 holds true, the sequence $p_n, p_n', \cdots, p_n^{(n)}$ obeys Descartes' rule in $(0,\infty)$. Since

$$H_k(p_n^{(q)};x) = W(p_n^{(q)},\ldots,p_n^{(q+k-1)};x),$$

then, by Theorem 2, $H_k(p_n^{(q)};x)$ does not vanish in $(0,\infty)$. Moreover, Theorem 2 implies also, that

$$sign\ H_k(p_n^{(q)};x) = sign\ W(p_n^{(n-k+1)},\ldots,p_n^{(n)};x) = sign\ H_k(p_n^{(n-k+1)};x) \quad for \quad x>0.$$

Observe that $H_k(p_n^{(n-k+1)};x)$ has zero units below the secondary diagonal and the units on this diagonal are positive numbers. Hence

$$sign\ H_k(p_n^{(q)};x) = (-1)^{k(k-1)/2} \quad for \quad x>0.$$

Now, letting n to diverge, we obtain

$$(-1)^{k(k-1)/2}\ H_k(\varphi^{(q)};x) \geq 0 \quad for \quad x>0.$$

Since $H_k(\varphi^{(q)};x)$ is a continuous function of x in a neighbourhood of the origin, the latter equality imlies

$$(-1)^{k(k-1)/2}\ H_k(\varphi^{(q)};x) \geq 0 \quad for \quad x \geq 0.$$

The case when $m \neq 0$ or $\varphi = cx^m e^{\sigma x}$ can be easily reduced to the previous consideration by continuos arguments. Although the associated Jensen polynomials may have multiple zeros, they are still real and nonnegative. Denote by $x_1^0, x_2^0, \ldots, x_n^0$ the zeros of $p_n(x) = g_n(z/n)$, where every zero appears as many times as its multiplicity. Set $r_n(x) = n^{-n}\gamma_n(x-x_1)\cdots(x-x_n)$. Note that r_n converges locally uniformly to p_n as $x_1, x_2, \ldots, x_n$ tend to $x_1^0, x_2^0, \ldots, x_n^0$ in such a way that $x_1, x_2, \ldots, x_n$ remain negative and distinct. Thus, for any nonnegative integer q, $r_n^{(q)}$ also converges locally uniformly to $p_n^{(q)}$. We have already proved that when $n \geq q+k-1$

$$sign\ H_k(r_n^{(q)};x) = (-1)^{k(k-1)/2} \quad for \quad x>0.$$

Letting the zeros of r_n to converge to the zeros of p_n and then n to diverge we get again

$$(-1)^{k(k-1)/2}\ H_k(\varphi^{(q)};x) \geq 0 \quad for \quad x \geq 0.$$

In order to complete the proof it remains only to note that the statement of the theorem follows by the same arguments in the case when φ is a polynomial.

References

[1] R. Boas, "Entire Functions," Academic Press, New York, 1954.

[2] S. Bochner, Über Sturm-Liouvillesche Polynomsysteme, *Math. Z.* **29**(1929), 730-736.

[3] T. S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon and Breach, New York, 1978.

[4] T. Craven and G. Csordas, Jensen polynomials and the Turán and Laguerre inequalities, *Pacific J. Math.* **136**(1989), 241-260.

[5] G. Csordas, T. S. Norfolk and R. S. Varga, The Riemann hypothesis and the Turán inequalities, *Trans. Amer. Math. Soc.* **296**(1986), 521-541.

[6] G. Csordas and R. S. Varga, Necessary and sufficient conditions and the Riemann hypothesis, *Adv. Appl. Math.* **11**(1990), 328-357.

[7] G. Csordas and J. Williamson, The zeros of Jensen polynomials are simple, *Proc. Amer. Math. Soc.* **49**(1975), 263-264.

[8] S. Karlin, "Total Positivity, vol. I," Stanford University Press, Stanford, California, 1968.

[9] E. Laguerre, "Oeuvres, vol. I," Paris, 1898.

[10] M. Marden, A rule of signs involving certain orthogonal polynomials, *Ann. Math.* **33**(1932), 118-124.

[11] N. Obrechkoff, On the distribution of zeros of algebraic equations, *Annuaire Univ. Sofia* **15-16**(1918-1919, 1919-1920), 1-14, 1-11, 1-4 (in Bulgarian).

[12] N. Obrechkoff, On the roots of algebraic equations, *Annuaire Univ. Sofia* **19**(1922-1923), 43-76 (in Bulgarian).

[13] N. Obrechkoff, Über die Wurzeln von algebraischen Gleichungen, *Jber. Deutschen Math. Ver.* **33**(1924), 52-64.

[14] N. Obrechkoff, "Zeros of Polynomials," Publ. Bulg. Acad. Sci., Sofia, 1963 (in Bulgarian); German translation: "Verteilung und Berchnung der Nullstellen Reeller Polynome," VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[15] G. Pólya, Über Annäherung durch Polynome mit lauter reelen Wurzeln, *Rend. Circ. Matem. Palermo* **36**(1913), 279-295.

[16] G. Pólya, Über die algebraisch-funktiontheoretischen Untersuchungen von J.L.W.V.Jensen, *Klg. Danske Vid. Sel. Math.-Fys. Medd.* **7**(1927), 3-33.

[17] G. Pólya and J. Schur, Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen, *J. Reine Angew. Math.* **144**(1944), 89-113.

[18] G. Pólya and G. Szegő, "Problems and Theorems in Analysis, vol. II," Springer-Verlag, Berlin, 1976.

[19] I.J. Schoenberg, Zur Abzählung der reelen Wurzeln algebraischer Gleichungen, *Math.Z.* 38(1934), 546-564.

[20] R.S.Varga, “Scientific Computation on Mathematical Problems and Conjectures,” Regional Conf. Ser. Appl. Math. Vol. 60, SIAM, Philadelphia, PA, 1990.