Inequalities for zeros of associated polynomials and derivatives of orthogonal polynomials

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Abstract

It is well-known and easy to see that the zeros of both the associated polynomial and the derivative of an orthogonal polynomial $p_n(x)$ interlace with the zeros of $p_n(x)$ itself. The natural question of how these zeros interlace is under discussion. We give a sufficient condition for the mutual location of k-th, $1 \le k \le n-1$, zeros of the associated polynomial and the derivative of an orthogonal polynomial in terms of inequalities for the corresponding Cotes numbers. Applications to the zeros of the associated polynomials and the derivatives of the classical orthogonal polynomials are provided. Various inequalities for zeros of higher order associated polynomials and higher order derivatives of orthogonal polynomials are proved. The results involve both classical and discrete orthogonal polynomials, where, in the discrete case, the differential operator is substituted by the difference operator.

Key words: Classical orthogonal polynomials, discrete orthogonal polynomials, associated polynomials, interlacing, Cotes numbers.

1 Introduction and statement of the general problem

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials, orthogonal on the interval (a, b) with respect to the positive measure $d\mu(x)$. As it is well-known, the zeros

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$x_1, \ldots, x_n$ of $p_n(x)$ are all real, distinct and belong to (a, b). In what follows we suppose that $x_1, \ldots, x_n$ are enumerated in increasing order. By Rolle's theorem, the zeros $\xi_1, \ldots, \xi_{n-1}$ of $Dp_n(x) := p_n'(x)$ are also real and interlace with the zeros of $p_n(x)$, i.e. $x_k < \xi_k < x_{k+1},\ k = 1, \ldots, n-1$. Denote by $Ap_n(x)$ the associated polynomial of degree $n-1$,

$$Ap_n(x) = \frac{1}{\mu_0} \int_a^b \frac{p_n(x) - p_n(t)}{x - t} d\mu(t), \quad \mu_0 = \int_a^b d\mu(t).$$

It is well-known and easy to see that $Ap_n(x)$ has $n-1$ real simple zeros $z_1, \ldots, z_{n-1}$, which also interlace with the zeros of $p_n(x)$, $x_k < z_k < x_{k+1},\ k = 1, \ldots, n-1$. Then the natural question about the mutual location of z_k and $\xi_k,\ k = 1, \ldots, n-1$, arises, and for families of classical continuous orthogonal polynomials it was posed in a form of conjecture by the second of us in [11].

Elbert and Laforgia [3] proved the inequalities $z_k > \xi_k$ for the positive zeros of the associated polynomial and the derivative of the Hermite polynomial $H_n(x)$ and of the ultraspherical polynomial $C_n^\lambda(x)$ in the case $\lambda > 0$. They pointed out in Remark 2 of [3] that $\xi_k < z_k$ for all the zeros of the associated polynomial and the derivative of the Jacobi polynomial $P_n^{\alpha,\beta}(x)$ provided

$$(\alpha, \beta) \in \Omega := \{\alpha \geq -1/2,\ -1 < \beta \leq -1/2,\ (\alpha, \beta) \neq (-1/2, -1/2)\},$$

and that the opposite inequalities $\xi_k > z_k$ hold if (α, β) is in the semi-infinite strip $\hat{\Omega}$ of the (α, β)-plane that is symmetric to Ω with respect to the line $\alpha = \beta$. Note that earlier Peherstorfer [8] established the inequalities $\xi_k > z_k$ for the positive zeros of associated polynomial and derivative of the ultraspherical polynomial $C_n^\lambda(x)$ in the case $-1/2 < \lambda < 0$. It is worth mentioning that for $\lambda = 0$ both the associated polynomial and the derivative of the Chebyshev polynomial $T_n(x)$ coincide with the Chebyshev polynomial of the second kind $U_{n-1}(x)$ and then the equalities $\xi_k = z_k$ hold for all k and n if $\lambda = 0$. Thus the problem for the Hermite and Gegenbauer polynomials has been settled completely. Peherstorfer and Schmuckenschläger [9] investigated in details the Jacobi case. Unaware of the above mentioned remark of [3], they proved the inequalities $\xi_k < z_k$ for a subregion of Ω and conjectured the above mentioned inequalities of Elbert and Laforgia. Peherstorfer and Schmuckenschläger [9] established an important fact: if (α, β) is in the complement of $\Omega \bigcup \hat{\Omega}$ to $\{\alpha > -1, \beta > -1\}$, then for any $n > 3$ there exist some indices $k,\ 1 \leq k \leq n-1$, for which $\xi_k < z_k$ and other indices k for which the opposite inequalities are true. As it was mentioned in [3], the corresponding inequalities for the zeros of associated polynomial and the derivative of the Laguerre polynomial $L_n^{(\alpha)}(x)$ can be obtained as a limit of the inequalities for the Jacobi case and formally, we can state that $\xi_k < z_k$ for all n and $k = 1, \ldots, n-1$ if $-1 < \alpha \leq -1/2$.

In the next section we prove that certain inequalities for the Cotes numbers imply the inequalities $\xi_k < z_k$. Then all the above stated results become consequences of monotonicity properties of the Cotes numbers corresponding to the classical orthogonal polynomials.

We also raise and investigate partially the more general question of zeros of higher order associated polynomials and higher order derivatives (differences) of orthogonal polynomials. Let the sequence $\{p_n\}_{n=0}^{\infty}$ be generated by the recurrence relation

$$\begin{aligned} p_{-1}(x) &= 0, \\ p_0(x) &= 1, \\ xp_n(x) &= a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad n \geq 0. \end{aligned}$$

For any positive integer l, the sequence $\{A^lp_n(x)\}_{n=1}^{\infty}$ of the l-th associated polynomials with the orthogonal polynomials $p_n(x)$ is generated by

$$\begin{aligned} A^lp_{l-1}(x) &= 0, \\ A^lp_l(x) &= 1, \\ x\ A^lp_n(x) &= a_{n+1}\ A^lp_{n+1}(x) + b_n\ A^lp_n(x) + c_n\ A^lp_{n-1}(x), \quad n \geq l. \end{aligned}$$

We avoid the more common denotation for associated polynomials (as on p. 87 in Chihara's book [1]) because of the confusion with the denotation for derivatives of the corresponding orthogonal polynomials. We have chosen this new denotation in order to emphasize the fact that we consider the "association" A as an operator which decreases by one the degree of orthogonal polynomials. Thus $A^lp_n(x)$ is a polynomial of degree $n-l$. We shall denote by $D^lp_n(x)$ the l-th derivative of $p_n(x)$. The operators A^l and D^l obey some common properties. For any fixed n both $\{D^lp_n(x)\}_{l=0}^{n}$ and $\{A^lp_n(x)\}_{l=0}^{n}$ are Sturm sequences in (a,b). Recall that a sequence of polynomials $\{q_j(x)\}_{j=0}^{n}$ with $deg(q_j(x)) = n-j,\ j = 0,\dots,n$, is a Sturm sequence in an open interval I if

(i) each $q_j(x)$ has exactly $n-j$ simple zeros in I,

(ii) for each $j,\ 1 \leq j \leq n-1$, the zeros of $q_j(x)$ and $q_{j+1}(x)$ strictly interlace.

See (4) on p. 45 in Szegő's book [14] for a formally different but equivalent definition. The fact that $\{A^lp_n(x)\}_{l=0}^{n}$ is a Sturm sequence can be easily proved by two simple observations. The above recurrence relations show that the zeros of $A^lp_n(x)$ are the eigenvalues of the Jacobi matrix obtained by deleting the first l rows and the first l columns in the Jacobi matrix associated with the polynomial $p_n(x)$. Then we can apply the interlacing eigenvalues theorem for bordered matrices (see Theorem 4.3.8 in [5] and its proof). Rolle's theorem implies immediately that $\{D^lp_n(x)\}_{l=0}^{n}$ is a Sturm sequence.

Observe that we can apply the operators $D^m A^l,\ m, l \geq 0$, to any sequence of orthogonal polynomials and, for any fixed l and m with $l + m = n$, the sequence $p_n, Ap_n, \ldots, A^l p_n, DA^l p_n, \ldots, D^m A^l p_n$ is a Sturm sequence in (a, b). The operators $A^l D^m,\ m, l \geq 0$, can not be applied to any sequence of orthogonal polynomials but $A^l D^m p_n$ are certainly well-defined when p_n is a sequence of classical orthogonal polynomials. Moreover, in the latter case, if l and m, $l + m = n$ are fixed, then $p_n, Dp_n, \ldots, D^m p_n, AD^m p_n, \ldots, A^l D^m p_n$ forms a Sturm sequence in (a, b).

These observations already suggest the more general and more interesting question about the mutual location of k-th zeros of $D^{m_1} A^{l_1} p_n(x)$ and of $D^{m_2} A^{l_2} p_n(x)$ provided $l_1 + m_1 = l_2 + m_2 = r$, where $r,\ 1 \leq r \leq n - 1$ is a fixed integer. It is of interest also to investigate the mutual location of the k-th zeros of $A^{l_1} D^{m_1} p_n(x)$ and of $A^{l_2} D^{m_2} p_n(x)$ under the same requirements on l_1, m_1, l_2 and m_2 when $\{p_n(x)\}$ is a sequence of classical orthogonal polynomials. The natural analogue of these two problems to the case when $\{p_n(x)\}$ is a sequence of discrete orthogonal polynomials is to simply substitute the differential operator D by the difference operator Δ,

$$\Delta f(x) = f(x+1) - f(x).$$

In Section 3 we prove various results on the largest zeros of the polynomials $A^l D^m p_n(x)$ and $D^m A^l p_n(x)$ for different values of the nonnegative integers l and m when $\{p_n(x)\}$ is a sequence of classical orthogonal polynomials with respect to a continuous measure. In the case of classical discrete orthogonal polynomials $\{p_n(x)\}$ we compare the largest zeros of $A^l \Delta^m p_n(x)$ and $\Delta^m A^l p_n(x)$.

2 Inequalities for all the zeros of the associated polynomial and the derivative of an orthogonal polynomial

Denote by π_n the space of the real algebraic polynomials of degree not exceeding n. Let $\lambda_j,\ j = 1, \ldots, n,$ be the Cotes numbers of the n-node Gaussian quadrature formula, corresponding to the positive measure $d\mu(x)$,

$$\int_a^b f(x) d\mu(x) = \sum_{j=1}^{n} \lambda_j f(x_j), \quad f \in \pi_{2n-1}.$$

We need to recall the basic fact that all the Cotes numbers $\lambda_j,\ j = 1, \ldots, n,$ are strictly positive. Our main result in this section follows.

Theorem 1 *Let, for some $k,\ 2 \leq k \leq n$, the inequalities*

$$\lambda_1 \leq \lambda_k \text{ for } j = 1, \ldots, k-1 \text{ and } \lambda_k \leq \lambda_j \text{ for } j = k+1, \ldots, n, \quad (2.\ 1)$$

hold, where the second set of inequalities is empty when $k = n$. Then $z_{k-1} \leq \xi_{k-1}$ and $z_k \leq \xi_k$. If at least one of the inequalities (2. 1) *is strict, then $z_{k-1} < \xi_{k-1}$ and $z_k < \xi_k$. The opposite inequalities for λ_j versus λ_k, $j \neq k$, imply the opposite inequalities for z_{k-1} versus ξ_{k-1} and z_k versus ξ_k.*

Proof: It is easy to see that every function of the form

$$f(x) = \sum_{j=1}^{n} \frac{b_j}{x - x_j}, \quad b_j > 0,$$

is strictly decreasing in every interval (x_j, x_{j+1}), $k = j, \ldots, n-1$, and has a unique zero there. When $b_j = \lambda_j$ the function $f(x)$ coincides with the partial fraction decomposition of $g(x) := Ap_n(x)/p_n(x)$ and $f(x)$ reduces to $h(x) := Dp_n(x)/p_n(x)$ when $b_j = 1$. Consider the function

$$g(x) - \lambda_k h(x) = \frac{Ap_n(x)}{p_n(x)} - \lambda_k \frac{Dp_n(x)}{p_n(x)} = \sum_{j=1}^{k-1} \frac{\lambda_j - \lambda_k}{x - x_j} + \sum_{j=k+1}^{n} \frac{\lambda_j - \lambda_k}{x - x_j}$$

in the intervals (x_k, x_{k+1}) and (x_k, x_{k+1}). Since the denominators of the quotients in the first sum are positive and in the second sum negative, inequalities (2. 1) immediately yield

$$g(x) - \lambda_k h(x) \leq 0 \quad \text{for} \;\; x \in (x_{k-1}, x_{k+1}).$$

Hence the only zero z_{k-1} of $g(x)$ in (x_{k-1}, x_k) is not greater than the only zero ξ_{k-1} of $h(x)$ in the same interval. Similarly, the only zero z_k of $g(x)$ in (x_k, x_{k+1}) is not greater than the only zero ξ_k of $h(x)$ in the same interval. Moreover, if at least one of the inequalities (2. 1) is strict, then we have $g(x) < \lambda_k h(x)$ for $x \in (x_{k-1}, x_{k+1})$, which implies the strict inequalities $z_{k-1} < \xi_{k-1}$ and $z_k < \xi_k$. The statement concerning the inverse inequalities is now obvious.

A similar result can be proven for the positive zeros of the associated polynomial and the derivative of a symmetric orthogonal polynomial. Let $\{p_n\}$ be a sequence of polynomials which are orthogonal with respect to an even weight function on a symmetric interval with respect to the origin. Then, as it is well-known, the polynomials p_n of even (odd) degree are even (odd) functions. Hence their derivatives as well as the associated polynomials obey the same property. In particular, for the zeros of these polynomials we have: $x_j = -x_{n+1-j}$ for $j = 1, \ldots, n$, and $z_j = -z_{n-j}$, $\xi_j = -\xi_{n-j}$ for $j = 1, \ldots, n-1$. Moreover, the corresponding Cotes numbers satisfy $\lambda_j = \lambda_{n+1-j}$. Then the above difference $g(x) - \lambda_k h(x)$ takes the form

$$g(x) - \lambda_k h(x) = 2x \left\{ \sum_{j=[(n+3)/2]}^{k-1} \frac{\lambda_j - \lambda_k}{x^2 - x_j^2} + \sum_{j=k+1}^{n} \frac{\lambda_j - \lambda_k}{x^2 - x_j^2} \right\}.$$

Thus we obtain the following result for the positive zeros of $Ap_n(x)$ and $Dp_n(x)$.

Corollary 1 *Let $\{p_n\}$ be a sequence of polynomials which are orthogonal with respect to a even weight function on a symmetric interval with respect to the origin. Let, for some k, $[(n+5)/2] \le k \le n$, the inequalities*

$$\lambda_j \le \lambda_k, \text{ for } j = [\frac{n+3}{2}], \ldots, k-1 \text{ and } \lambda_k \le \lambda_j \text{ for } j = k+1, \ldots, n, \quad (2.\ 2)$$

hold, where the second subset of inequalities is empty when $k = n$. Then $z_{k-1} \le \xi_{k-1}$ and $z_k \le \xi_k$. If at least one of the inequalities (2. 2) *is strict, then $z_{k-1} < \xi_{k-1}$ and $z_k < \xi_k$. The opposite inequalities for λ_j versus λ_k, $j \ne k$, imply the opposite inequalities for z_{k-1} versus ξ_{k-1} and z_k versus ξ_k.*

Now all the results concerning the classical orthogonal polynomials, mentioned in the first section, follow from the monotonicity properties of the corresponding Cotes numbers (see Section 15.3 in Szegő [14]):

(1) Hermite polynomials $H_n(x)$: $\lambda_{[(n+3)/2]} > \lambda_{[(n+5)/2]} > \cdots > \lambda_n$,
(2) Gegenbauer polynomials $C_n^\lambda(x)$: $\lambda_{[(n+3)/2]} > \lambda_{[(n+5)/2]} > \cdots > \lambda_n$ if $\lambda > 0$ and $\lambda_{[(n+3)/2]} < \lambda_{[(n+5)/2]} < \cdots \lambda_n$ if $-1/2 < \lambda < 0$.
(3) Laguerre polynomials $L_n^{(\alpha)}(x)$: $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ if $-1 < \alpha \le -1/2$.
(4) Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$: $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ if $(\alpha,\beta) \in \Omega$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ if $(\alpha,\beta) \in \hat{\Omega}$.

Note that these inequalities are proved by using Sonin's (also called Pólya-Sonin's) theorem (see Theorem 7.31.1 in [14] and Theorem 8.18 in [4]) on the behaviour of the local extremal values of a solution of a second order differential equation. Elbert and Laforgia [3] used a somehow similar idea but they neither formulated explicitly general results of the nature of the above Theorem 1 and Corollary 1, nor referred to Sonin's theorem.

Now we prove that the zeros of $D^m p_n(x)$ and $D^{m-1}Ap_n(x)$ 'inherit' the inequalities between the zeros of $Dp_n(x)$ and $Ap_n(x)$. In what follows we denote by $x_k(A^l D^m p_n)$ and $x_k(D^m A^l p_n)$ the k-th zero, in increasing order, of the $n-m-l$ degree polynomials $A^l D^m p_n(x)$ and $D^m A^l p_n(x)$, respectively.

Theorem 2 *Let n and m be any positive integers with $m \le n-1$. Then the inequalities hold:*

(a) $x_k(D^m H_n) < x_k(D^{m-1}AH_n)$ *for* $k = [(n-m+3)/2], \ldots, n$.

(b) $x_k(D^m C_n^\lambda) < x_k(D^{m-1}AC_n^\lambda)$ *if* $\lambda > 0$ *and* $x_k(D^m C_n^\lambda) > x_k(D^{m-1}AC_n^\lambda)$ *if* $-1/2 < \lambda < 0$, *for* $k = [(n-m+3)/2], \ldots, n$.

(c) $x_k(D^m L_n^{(\alpha)}) < x_k(D^{m-1} A L_n^{(\alpha)})$ *for* $-1 < \alpha \le -1/2$ *and* $k = 1, \ldots, n-m$.

(d) $x_k(D^m P_n^{(\alpha,\beta)}) < x_k(D^{m-1} A P_n^{(\alpha,\beta)})$ *if* $(\alpha, \beta) \in \Omega$ *and* $x_k(D^m P_n^{(\alpha,\beta)}) > x_k(D^{m-1} A P_n^{(\alpha,\beta)})$ *if* $(\alpha, \beta) \in \hat{\Omega}$*, for* $k = 1, \ldots, n-m$.

Proof. We need only to apply a theorem of V. Markov (see Lemma 2.7.1 in Rivlin[10]) and a version of it for the positive zeros of even (odd) polynomials proved by the first author (see Lemma 1 and Corollary 2 in [2]) to the inequalities between the zeros of the $Dp_n(x)$ and $Ap_n(x)$ established by Elbert and Laforgia and by Peherstorfer and Schmuckenschläger.

V. Markov's theorem reads as follows: If $p(x) = (x - a_1) \cdots (x - a_n)$ and $q(x) = (x - b_1) \cdots (x - b_n)$, where $a_1 > b_1 > a_2 > b_2 > \cdots > a_n > b_n$, then, if $t_1, \cdots, t_{n-1}$ are the zeros of $p'(x)$ and $u_1, \cdots, u_{n-1}$ are the zeros of $q'(x)$ (each set arranged in decreasing order) we have $t_1 > u_1 > t_2 > u_2 > \cdots > t_{n-1} > u_{n-1}$.

The corresponding result for the zeros of even (odd) polynomials and the zeros of their derivatives proved in [2] reads as follows: Let $q(x)$ be a polynomial of degree $n \ge 3$ with distinct real zeros. Suppose that $q(x)$ is even (odd) if n is even (odd). Then every positive zero of $q'(x)$ is an increasing function of any positive zero of $q(x)$.

3 Inequalities for the largest zeros of high order associated polynomials and high order derivatives (differences) of orthogonal polynomials

As was already mention, Peherstorfer and Schmuckenschläger [9] proved that for $\alpha > -1/2, \beta > -1/2$ the mutual location of the zeros ξ_k and z_k depends essentially on the index k. In this section we are interested on the mutual location of the largest zeros of the polynomials $A^{l_1} D^{m_1} p_n(x)$ and $A^{l_2} D^{m_2} p_n(x)$, $l_1 + m_1 = l_2 + m_2 = r$, when $\{p_n(x)\}$ is a sequence of classical continuous orthogonal polynomials. In the case when $\{p_n(x)\}$ is a sequence of classical discrete orthogonal polynomials we compare the largest zeros of $A^{l_1} \Delta^{m_1} p_n(x)$ and $A^{l_2} \Delta^{m_2} p_n(x)$

We apply a basic tool to investigate this question. It is based on the the above mentioned interpretation of the zeros of the polynomials under discussion as eigenvalues of Jacobi matrices. The zeros of $p_n(x)$ coincide with the eigenvalues of the $n \times n$ Jacobi matrix $J(p_n)$ whose diagonal elements are $a_{jj} = b_{j-1},\ j = 1, \ldots, n$ and its off-diagonal elements are $a_{j+1,j} = a_{j,j+1} = a_j,\ j = 1, \ldots, n-1$ and the zeros of $A^l p_n(x)$ coincide with the eigenvalues of the $(n-l) \times (n-l)$ Jacobi matrix $J(A^l p_n)$ whose non-zero elements are $a_{jj} = b_{j+l-1},\ j =$

$1, \ldots, n-l$ and $a_{j+1,j} = a_{j,j+1} = a_{j+l},\ j = 1, \ldots, n-l-1$. We apply then the Perron–Frobenius theorem for non-negative matrices (see Theorem 8.4.5 in [5]). In our case it is applicable and we can state that the inequality $\rho(B) \leq \rho(C)$ holds for the largest eigenvalues $\rho(B)$ and $\rho(C)$ of the non-negative matrices (matrices with non-negative elements) B and C provided $C - B$ is a non-negative matrix. Moreover, $\rho(B) < \rho(C)$ if $C - B$ is a non-negative matrix with at least one positive entry. To the best of our knowledge, Ismail [6] was the first to apply the Perron–Frobenius theorem to investigate the location and behaviour of largest zeros of orthogonal polynomials.

3.1 The largest zeros of $A^l D^m p_n(x)$ and of $A^{l+m} p_n(x)$ for the classical orthogonal polynomials

In this subsection we shall establish inequalities between the largest zeros of the polynomials $A^{l_1} D^{m_1} p_n(x)$ and $A^{l_2} D^{m_2} p_n(x)$ when $p_n(x)$ is a classical orthogonal polynomials and $l_1 + m_1 = l_2 + m_2 = r$. Denote these zeros by $x_{n-r}(A^{l_1}\ D^{m_1} p_n)$ and $x_{n-r}(A^{l_2}\ D^{m_2} p_n)$.

Theorem 3 *Let n and r be any positive integers with $r \leq n-1$. If $l_1 + m_1 = l_2 + m_2 = r$ then the following inequalities for the largest zeros of $A^{l_1} D^{m_1} p_n(x)$ and of $A^{l_2} D^{m_2} p_n(x)$ hold:*

(a) $x_{n-r}(A^{l_1} D^{m_1} H_n) < x_{n-r}(A^{l_2} D^{m_2} H_n)$ if and only if $l_1 < l_2$.

(b) If $\lambda > 0$, then the inequalities $x_{n-r}(A^{l_1} D^{m_1} C_n^{\lambda}) < x_{n-r}(A^{l_2} D^{m_2} C_n^{\lambda})$ hold if and only if $l_1 < l_2$.

(c) If $\alpha > 0$, then the inequalities $x_{n-r}(A^{l_1} D^{m_1} L_n^{(\alpha)}) < x_{n-r}(A^{l_2} D^{m_2} L_n^{(\alpha)})$ hold if and only if $l_1 < l_2$.

(d) If $\alpha > \beta > -1$, then $x_{n-r}(A^{l_1} D^{m_1} P_n^{(\alpha,\beta)}) < x_{n-r}(A^{l_2} D^{m_2} P_n^{(\alpha,\beta)})$ if and only if $l_1 < l_2$.

Proof. The families of classical orthogonal polynomials are closed under differentiation. In particular, with the usual denotation $(c)_n$ for the Pochhammer symbol, the following identities hold (see formulae 4.21.7, 4.7.14, 5.1.14 and 5.5.10 in [14]):

$$\begin{aligned}
D^m P_n^{(\alpha,\beta)}(x) &= 2^{-m}(n+\alpha+\beta+1)_m P_{n-m}^{(\alpha+m,\beta+m)}(x), \\
D^m C_n^{(\lambda)}(x) &= 2^m \lambda^m C_{n-m}^{(\lambda+m)}(x), \\
D^m L_n^{(\alpha)}(x) &= (-1)^m L_{n-m}^{(\alpha+m)}(x), \\
D^m H_n(x) &= 2^m (n-m+1)_m H_{n-m}(x).
\end{aligned}$$

In what follows we shall denote by $b_j(p_n)$ and $a_j(p_n)$ the diagonal and the off-diagonal elements of the Jacobi matrix associated with $p_n(x)$.

For the Jacobi polynomials we have

$$b_j(P_n^{(\alpha,\beta)}) = \frac{\beta^2-\alpha^2}{(\alpha+\beta+2j+2)(\alpha+\beta+2j)}, \quad j=0,\dots,n-1,$$
$$a_j(P_n^{(\alpha,\beta)}) = \frac{2}{\alpha+\beta+2j}\sqrt{\frac{j(\alpha+\beta+j)(\alpha+j)(\beta+j)}{(\alpha+\beta+2j-1)(\alpha+\beta+2j+1)}},$$
$$j=1,\dots,n-1.$$

Then

$$b_j(D^m P_n^{(\alpha,\beta)}) = \frac{(\beta+m)^2-(\alpha+m)^2}{(\alpha+\beta+2m+2j+2)(\alpha+\beta+2m+2j)},$$
$$j=0,\dots,n-m-1,$$
$$a_j(D^m P_n^{(\alpha,\beta)}) = \frac{2}{\alpha+\beta+2m+2j} \times$$
$$\sqrt{\frac{j(\alpha+\beta+2m+j)(\alpha+m+j)(\beta+m+j)}{(\alpha+\beta+2m+2j-1)(\alpha+\beta+2m+2j+1)}},$$
$$j=1,\dots,n-m-1.$$

and, for $m+l=r$,

$$b_j(A^l D^m P_n^{(\alpha,\beta)}) = \frac{(\beta+m)^2-(\alpha+m)^2}{(\alpha+\beta+2(r+j+1))(\alpha+\beta+2(r+j))},$$
$$j=0,\dots,n-m-l-1,$$
$$a_j(A^l D^m P_n^{(\alpha,\beta)}) = \frac{2}{\alpha+\beta+2m+2l+2j} \times$$
$$\sqrt{\frac{(l+j)(\alpha+\beta+m+r+j)(\alpha+r+j)(\beta+r+j)}{(\alpha+\beta+2(r+j)-1)(\alpha+\beta+2(r+j)+1)}},$$
$$j=1,\dots,n-m-l-1.$$

Let $l_1+m_1=l_2+m_2=r$. Then, for $j=0,\dots,n-r-1$, we have

$$\begin{aligned}
&b_j(A^{l_1}D^{m_1}P_n^{(\alpha,\beta)}) - b_j(A^{l_2}D^{m_2}P_n^{(\alpha,\beta)})\\
&= \frac{(\beta+m_1)^2-(\alpha+m_1)^2-(\beta+m_2)^2+(\alpha+m_2)^2}{(\alpha+\beta+2r+2j+2)(\alpha+\beta+2r+2j)}\\
&= \frac{2(\beta-\alpha)(m_1-m_2)}{(\alpha+\beta+2r+2j-2)(\alpha+\beta+2r+2j)},
\end{aligned}$$

and, for $j = 1, \ldots, n-r-1$,

$$\begin{aligned}
&a_j(A^{l_1} D^{m_1} P_n^{(\alpha,\beta)}) - a_j(A^{l_2} D^{m_2} P_n^{(\alpha,\beta)}) \\
&= \frac{2}{\alpha+\beta+2r+2j}\sqrt{\frac{(\alpha+r+j)(\beta+r+j)}{(\alpha+\beta+2r+2j-1)(\alpha+\beta+2r+2j+1)}}\times \\
&\left\{\sqrt{c_j(l_1,m_1)} - \sqrt{c_j(l_2,m_2)}\right\},
\end{aligned}$$

where $c_j(l,m) := (j+l)(\alpha+\beta+r+j+m)$. Obviously $c_j(l,m) > 0$ for every positive integer r and

$$c_j(l_1,m_1) - c_j(l_2,m_2) = -(\alpha+\beta+m_1+m_2)(m_1-m_2)$$

for any index j.

Thus, for $\alpha \geq \beta$, the inequalities

$$\begin{aligned}
b_j(A^{l_1} D^{m_1} P_n^{(\alpha,\beta)}) &< b_j(A^{l_2} D^{m_2} P_n^{(\alpha,\beta)}), \quad j = 0, \ldots, n-r-1, \\
a_j(A^{l_1} D^{m_1} P_n^{(\alpha,\beta)}) &< a_j(A^{l_2} D^{m_2} P_n^{(\alpha,\beta)}), \quad j = 1, \ldots, n-r-1,
\end{aligned}$$

hold simultaneously if and only if $m_2 < m_1$ which is equivalent to $l_1 < l_2$.

For the Laguerre polynomials we have

$$\begin{aligned}
b_j(L_n^{(\alpha)}) &= \alpha+2j+1, \quad j = 0, \ldots, n-1, \\
a_j(L_n^{(\alpha)}) &= \sqrt{j(\alpha+j)}, \quad j = 1, \ldots, n-1.
\end{aligned}$$

Then

$$\begin{aligned}
b_j(D^m L_n^{(\alpha)}) &= \alpha+m+2j+1, \quad j = 0, \ldots, n-m-1, \\
a_j(D^m L_n^{(\alpha)}) &= \sqrt{j(\alpha+m+j)} \quad j = 1, \ldots, n-m-1.
\end{aligned}$$

and

$$\begin{aligned}
b_j(A^l D^m L_n^{(\alpha)}) &= \alpha+m+2l+2j+1, \quad j = 1, \ldots, n-m-l, \\
a_j(A^l D^m L_n^{(\alpha)}) &= \sqrt{(l+j)(\alpha+m+l+j)} \quad j = 1, \ldots, n-m-l-1.
\end{aligned}$$

If $l_1 + m_1 = l_2 + m_2 = r$ then

$$\begin{aligned}
b_j(A^{l_1} D^{m_1} L_n^{(\alpha)}) - b_j(A^{l_2} D^{m_2} L_n^{(\alpha)}) &= l_1 - l_2, \\
a_j(A^{l_1} D^{m_1} L_n^{(\alpha)}) - a_j(A^{l_1} D^{m_1} L_n^{(\alpha)}) &= \frac{\sqrt{\alpha+r+j}}{\sqrt{l_1+j}+\sqrt{l_2+j}}(l_1-l_2).
\end{aligned}$$

Therefore the largest zero of $A^{l_1}D^{m_1}L_n^{(\alpha)}(x)$ is greater (smaller) than the largest zeros of $A^{l_1}D^{m_1}L_n^{(\alpha)}(x)$ if and only if l_1 is greater (smaller) than l_2.

For the case of Hermite polynomials we have that all the matrices have zero diagonal elements and the corresponding off-diagonal elements are

$$a_j(H_n) = a_j(D^m H_n) = ((j+1)/2)^{1/2}, \quad a_j(A^l D^m H_n) = ((j+l+1)/2)^{1/2}.$$

Hence $a_j(A^{l_1}D^{m_1}H_n) < a_j(A^{l_2}D^{m_2}H_n)$ if and only if $l_1 < l_2$. This completes the proof of Theorem 3.

3.2 Inequalities for the largest zeros of $A^{l_1}\Delta^{m_1}p_n(x)$ and of $A^{l_2}\Delta^{m_2}p_n(x)$, $l_1+m_1 = l_2+m_2 = r$, for the classical discrete orthogonal polynomials

Define the difference operators Δ and ∇ by

$$\Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = \Delta f(x-1) = f(x) - f(x-1).$$

It is known that the classical discrete orthogonal polynomials which involve Charlier, Meixner, Krawtchouk and Hahn families satisfy a second order difference equations of the form (see [7])

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0. \qquad (3.\ 3)$$

More precisely, the Charlier polynomial $c_n^{\mu}(x)$, $\mu > 0$ satisfies (3. 3) with $\sigma(x) = x$, $\tau(x) = \mu - x$, $\lambda_n = n$, Meixner polynomial $m_n^{(\gamma,\mu)}(x)$, $0 < \mu < 1$, $0 < \gamma$ is a solution of (3. 3) with $\sigma(x) = x$, $\tau(x) = \gamma\mu - (1-\mu)x$, $\lambda_n = n(1-\mu)$, for the Krawtchouk polynomial $k_n^{(p)}(x)$, $p > 0$, $q > 0$, $p+q = 1$ we have $\sigma(x) = x$, $\tau(x) = (Np - x)/q$, $\lambda_n = n/q$ and Hahn polynomial $h_n^{(\alpha,\beta)}(x;N)$ satisfies (3. 3) with $\sigma(x) = x(N+\alpha-x)$, $\tau(x) = (\beta+1)(N-1)-(\alpha+\beta+2)x$, $\lambda_n = n(n+\alpha+\beta+1)$. The discrete Chebyshev polynomials $t_n(x)$ are defined by $t_n(x) = h_n^{(0,0)}(x;N)$

Theorem 4 *Let n and r be any positive integers with $r \le n-1$ and let l_1, l_2, m_1 and m_2 be such that $l_1+m_1 = l_2+m_2 = r$. Then the following inequalities between the largest zeros of $A^{l_1}\Delta^{m_1}p_n(x)$ and $A^{l_2}\Delta^{m_2}p_n(x)$ hold:*

(a) $x_{n-r}(A^{l_1}\Delta^{m_1}c_n^{(\mu)}) < x_{n-r}(A^{l_2}\Delta^{m_2}c_n^{(\mu)})$ if and only if $l_1 < l_2$.

(b) $x_{n-r}(A^{l_1}\Delta^{m_1}m_n^{(\gamma,\mu)}) < x_{n-r}(A^{l_2}\Delta^{m_2}m_n^{(\gamma,\mu)})$ if and only if $l_1 < l_2$.

(c) $x_{n-r}(A^{l_1}\Delta^{m_1}k_n^{(p)}) < x_{n-r}(A^{l_2}\Delta^{m_2}k_n^{(p)})$ if and only if $l_1 < l_2$.

(d) $x_{n-r}(A^{l_1}\Delta^{m_1}h_n^{(\alpha,\beta)}) < x_{n-r}(A^{l_2}\Delta^{m_2}h_n^{(\alpha,\beta)})$ if $\alpha > \beta$, $n < N$, $l_1 < l_2$ and $l_1+l_2 < 2r+\alpha+\beta$.

Proof. For the classical discrete orthogonal polynomials we have

$$
\begin{aligned}
\Delta^m c_n^{(\mu)}(x) &= \mu^{-m}(-n)_m c_{n-m}^{(\mu)}(x),\\
\Delta^m m_n^{(\gamma,\mu)}(x) &= ((1-\mu)/\mu)^{-m}(-n)_m m_{n-m}^{(\gamma+m,\mu)}(x),\\
\Delta^m k_n^{(p)}(x,N) &= k_{n-m}^{(p)}(x,N-m),\\
\Delta^m h_n^{(\alpha,\beta)}(x,N) &= (n+\alpha+\beta+1)_m h_{n-m}^{(\alpha+m,\beta+m)}(x,N-m).
\end{aligned}
$$

The elements of the Jacobi matrix associated with the classical discrete orthogonal polynomials are

$$b_j(c_n^{(\mu)}(x)) = j+\mu, \quad a_j(c_n^{(\mu)}(x)) = \sqrt{j\mu},$$

$$
\begin{aligned}
b_j(m_n^{(\gamma,\mu)}(x)) = \frac{j+\mu(j+\gamma)}{1-\mu}, &\quad a_j(m_n^{(\gamma,\mu)}(x)) = \frac{\sqrt{j\mu(j+\gamma-1)}}{1-\mu},\\
b_j(k_n^{(p)}(x,N)) = j+p(N-2j), &\quad a_j(k_n^{(p)}(x,N)) = \sqrt{jpq(N-j+1)},
\end{aligned}
$$

$$
\begin{aligned}
b_j(h_n^{(\alpha,\beta)}(x,N)) &= \frac{j(2N+\alpha-\beta-2)(\alpha+\beta+j+1)}{(\alpha+\beta+2j)(\alpha+\beta+2j+2)}\\
&\quad+\frac{(N-1)(\alpha+\beta)(\beta+1)}{(\alpha+\beta+2j)(\alpha+\beta+2j+2)},\\
a_j^2(h_n^{(\alpha,\beta)}(x,N)) &= \frac{j(j+\alpha)(j+\beta)(j+\alpha+\beta)(j+\alpha+\beta+N)(N-j)}{(2j+\alpha+\beta-1)(2j+\alpha+\beta)^2(2j+\alpha+\beta+1)}.
\end{aligned}
$$

Then the non-zero elements of the Jacobi matrices associated with $\Delta^m c_n^{(\mu)}(x)$ and with $A^l\Delta^m c_n^{(\mu)}(x)$ are defined by

$$
\begin{aligned}
b_j(\Delta^m c_n^{(\mu)}(x)) = j+\mu, &\quad a_j(\Delta^m c_n^{(\mu)}(x)) = \sqrt{j\mu},\\
b_j(A^l\Delta^m c_n^{(\mu)}(x)) = j+\mu+l, &\quad a_j(A^l\Delta^m c_n^{(\mu)}(x)) = \sqrt{(j+l)\mu}.
\end{aligned}
$$

For $l_1+m_1 = l_2+m_2 = r$ both $b_j(A^{l_1}\Delta^{m_1}c_n^{(\mu)}(x)) - b_j(A^{l_2}\Delta^{m_2}c_n^{(\mu)}(x))$ and $a_j(A^{l_1}\Delta^{m_1}c_n^{(\mu)}(x)) - a_j(A^{l_2}\Delta^{m_2}c_n^{(\mu)}(x))$ are positive (negative) if and only if l_1-l_2 is positive (negative).

The non-zero elements of the Jacobi matrices associated with $\Delta^m m_n^{(\gamma,\mu)}(x)$ and with $A^l\Delta^m m_n^{(\gamma,\mu)}(x)$ are defined by

$$
\begin{aligned}
b_j(\Delta^m m_n^{(\gamma,\mu)}(x)) &= \frac{j+\mu(j+\gamma+m)}{1-\mu},\\
a_j(\Delta^m m_n^{(\gamma,\mu)}(x)) &= \frac{(j\mu(j+\gamma+m-1))^{1/2}}{1-\mu},\\
b_j(A^l\Delta^m m_n^{(\gamma,\mu)}(x)) &= \frac{j+l+\mu(j+\gamma+m+l)}{1-\mu},\\
a_j(A^l\Delta^m m_n^{(\gamma,\mu)}(x)) &= \frac{((j+l)\mu(j+\gamma+m+l-1))^{1/2}}{1-\mu}.
\end{aligned}
$$

If $l_1+m_1=l_2+m_2=r$ then both $b_j(A^{l_1}\Delta^{m_1}m_n^{(\gamma,\mu)}(x))-b_j(A^{l_2}\Delta^{m_2}m_n^{(\gamma,\mu)}(x))$ and $a_j(A^{l_1}\Delta^{m_1}m_n^{(\gamma,\mu)}(x))-a_j(A^{l_2}\Delta^{m_2}m_n^{(\gamma,\mu)}(x))$ are positive (negative) if and only if l_1-l_2 is positive (negative).

The Jacobi matrices associated with $\Delta^m k_n^{(p)}(x,N)$ and $A^l\Delta^m k_n^{(p)}(x,N)$ are determined by

$$
\begin{aligned}
b_j(\Delta^m k_n^{(p)}(x,N)) &= j+p(N-2j-m),\\
a_j(\Delta^m k_n^{(p)}(x,N)) &= (jpq(N-m-j+1))^{1/2},\\
b_j(A^l\Delta^m k_n^{(p)}(x,N)) &= j+l+p(N-2(j+l)-m),\\
a_j(A^l\Delta^m k_n^{(p)}(x,N)) &= ((j+l)pq(N-m-j-l+1))^{1/2}.
\end{aligned}
$$

For $l_1+m_1=l_2+m_2=r$ we have

$$
b_j(A^{l_1}\Delta^{m_1}k_n^{(p)}(x,N))-b_j(A^{l_2}\Delta^{m_2}k_n^{(p)}(x,N))=p(m_2-m_1+l_2-l_1)=0.
$$

Under the same requirements on l_1, m_1, l_2 and m_2 the difference

$$
a_j(A^{l_1}\Delta^{m_1}k_n^{(p)}(x,N))-a_j(A^{l_2}\Delta^{m_2}k_n^{(p)}(x,N))
$$

is positive (negative) if and only if l_1-l_2 is positive (negative).

In the case of Hahn polynomials lengthly but straightforward computations show that

$$
\begin{aligned}
&(\alpha+\beta+2(r+j))(\alpha+\beta+2(r+j))b_j(A^l\Delta^m h_n^{(\alpha,\beta)})\\
&=(j+l)(\alpha+\beta+2r-l+1)(2(N+l-r-1)+\alpha-\beta)\,l\\
&+(\alpha+\beta+2(r-l))(N+l-r-1)(r-l+\beta+1)
\end{aligned}
$$

and

$$\begin{aligned}
&a_j^2(A^l\Delta^m h_n^{(\alpha,\beta)})\times \\
&(2(j+r)+\alpha+\beta-1)(2(j+r)+\alpha+\beta)^2(2(j+r)+\alpha+\beta+1) \\
&=(j+\alpha+r)(j+\beta+r)(j+\alpha+\beta+N+r)\times \\
&(N-r-j)(j+l)(j+\alpha+\beta+2r-l)
\end{aligned}$$

Then

$$\begin{aligned}
&(\alpha+\beta+2(r+j))(\alpha+\beta+2(r+j))\frac{b_j(A^{l_1}\Delta^{m_1}h_n^{(\alpha,\beta)})-b_j(A^{l_2}\Delta^{m_2}h_n^{(\alpha,\beta)})}{l_1-l_2} \\
&=2r^2+2(\alpha+\beta+2j+1)r+2j^2+2(\alpha+\beta+1)j+(\alpha+\beta)(\alpha+1) \\
&+N(\alpha-\beta).
\end{aligned}$$

Obviously the latter expression is positive if $\alpha>\beta$.

Similarly, we can conclude that the sign of $a_j^2(A^{l_1}\Delta^{m_1}h_n^{(\alpha,\beta)})-a_j^2(A^{l_2}\Delta^{m_2}h_n^{(\alpha,\beta)})$ is the same as the sign of

$$(l_1-l_2)(N-r-j)(\alpha+\beta+2r-l_1-l_2).$$

The proof of Theorem 4 is complete.

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