TWO NEW CONJECTURES CONCERNING POSITIVE JACOBI POLYNOMIAL SUMS

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Abstract. A refinement of a conjecture of Gasper concerning the values of (α,β), $-1/2<\beta<0$, $-1<\alpha+\beta<0$, for which the inequalities $\sum_{k=0}^{n} P_k^{(\alpha,\beta)}(x)/P_k^{(\beta,\alpha)}(1)\ge 0$, $-1\le x\le 1$, $n=1,2,\dots$, hold is stated. An algorithm for checking the new conjecture using the package "Matematica" is provided. Numerical results in support of the conjecture and an idea for proving it are given.

1. Introduction

The Jacobi polynomials are defined in terms of the hypergeometric function ${}_2F_1$ by

$$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_n}{n!}\,{}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;(1-x)/2),$$

where $(a)_k=\Gamma(a+k)/\Gamma(a)$ is the Pochhamer symbol and

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}.$$

Various special cases of the inequalities

$$(1)\qquad S_n^{(\alpha,\beta)}(x):=\sum_{k=0}^{n}P_k^{(\alpha,\beta)}(x)/P_k^{(\beta,\alpha)}(1)\ge 0,\quad -1\le x\le 1,\quad n=1,2,\dots$$

have been proved. Fejér [11, 12] was the first to establish inequalities of this from for $\alpha=1/2$, $\beta=-1/2$ and for $\alpha=\beta=0$. Fejér conjectured that (1) hold also for $\alpha=\beta=1/2$ and his conjecture was proved independently by Jackson [16] and Gronwall [15]. Feldheim [13] proved (1) for $\alpha=\beta\ge 0$. Some special cases of these inequalities were considered by Askey [1, 2]. Askey and Gasper [4] proved (1) for $\beta\ge 0$, $\alpha+\beta\ge -2$. The impotrance of the latter result is justified by the fact that de Branges [7] used (1) for $\beta=0$, $\alpha=2,4,6,\dots$, in the final step of his proof of the celebrated Bieberbach conjecture. Gasper [14] proved the inequalities (1) for $\beta\ge -1/2$, $\alpha+\beta\ge 0$.

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Note that Bateman's integral formula (see Bateman [6])

$$\frac{P_n^{(\alpha-\mu,\beta+\mu)}(x)}{P_n^{(\beta+\mu,\alpha-\mu)}(1)} = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^{x} \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\beta,\alpha)}(1)} \frac{(1+t)^{\beta}}{(1+x)^{\beta+\mu}} (x-t)^{\mu-1} dt, \tag{2}$$

which holds for $\mu > 0$ and $\beta > -1$ implies:

Lemma 1. *If inequalities* (1) *hold for* (α, β)*, they hold for any* $(\alpha - \mu, \beta + \mu)$*,* $\mu > 0$*.*

If (1) *fail for some* (α, β)*, they fail for any* $(\alpha + \mu, \beta - \mu)$*,* $\mu > 0$*.*

On the other hand, $S_1^{(\alpha,\beta)}(x) = (\alpha+\beta+2)(1+x)/(2(\beta+1))$. Having in mind these observations, the above mentioned results of Askey and Gasper [4] and of Gasper [14] yield:

The inequalities (1) hold for $\alpha \le 0$, $\beta \ge max\{0, -\alpha-2\}$ and $\alpha \ge 0$, $\beta \ge max\{-1/2, -\alpha\}$.

The inequalities (1) fail for $\beta < max\{-1/2, -\alpha-2\}$.

In 1993 Askey [3] drew attention to (1) for the rest of the (α, β)-plane, nemely, for (α, β) in the parallelogram $D_1 = \{-1/2 \le \beta < 0, -2 \le \alpha+\beta < 0\}$. It was proved in [10] that (1) fail for $x = 1$ and for sufficiently large n, if $|\alpha - 3/2| - 1/2 \le \beta < 0$. The latter and Bateman's integral (2) disprove inequalities (1) for the left-hand half of D_1, at least for n large enough. Thus, the only region in the (α, β)-plane, for which the inequalities (1) are still to be proved or disproved is the parallelogram

$$D = \{(\alpha, \beta) \ : \ -1/2 < \beta < 0, \ -1 \le \alpha + \beta < 0\}.$$

On the other hand, (1) hold for the upper boundary $\{\beta = 0, \ \ -1 \le \alpha < 0\}$ and fail for the lower boundary $\{\beta = -1/2, \ \ -1/2 \le \alpha < 1/2\}$ of D. Hence, by Bateman's integral, for any $\theta \in (-1, 0)$ there exist an $(\alpha', \beta') \in D$ with $\alpha' + \beta' = \theta$, such that (1) hold for $\{\alpha + \beta = \theta, \ \ \beta \ge \beta'\}$ and fails for $\{\alpha + \beta = \theta, \ \ \beta < \beta'\}$. The curve, formed by the points (α', β') with this property will be denoted by γ. Denote by $J_\alpha(x)$ the Bessel function of the first kind with parameter α and let $j_{\alpha,2}$ be the second positive zero of $J_\alpha(x)$. The following conjecture for the curve γ is due to Gasper [14, p. 444]:

Conjecture 1. *The subregion* Δ *of* D*, for which the inequalities* (1) *hold is given by*

$$\Delta = \{(\alpha, \beta) \in D : \ \ \beta \ge \beta(\alpha), \ \ \textit{where} \int_0^{j_{\alpha,2}} t^{-\beta(\alpha)} J_\alpha(t) dt = 0\}. \tag{3}$$

It may be pointed out that Gasper's conjecture is equivalent to the statement that γ is given by

$$\{(\alpha, \beta(\alpha)) \in D : \ \ \int_0^{j_{\alpha,2}} t^{-\beta(\alpha)} J_\alpha(t) dt = 0\}.$$

The conjecture is based on the well-known formula (see (1.8) in [3])

$$\lim_{n\to\infty} \left(\frac{\theta}{n}\right)^{\alpha-\beta+1} \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(\cos(\theta/n))}{P_k^{(\beta,\alpha)}(1)} = 2^\alpha \Gamma(\beta+1) \int_0^\theta t^{-\beta} J_\alpha(t) dt, \ \ \beta < \alpha + 1, \tag{4}$$

and on the following theorem:

Theorem 1. *Let $-1 < \alpha < 1/2$ and $\beta > -1/2$. Then the inequality*

$$\int_0^\theta t^{-\beta} J_\alpha(t)dt \geq 0,$$

hold for any nonnegative θ if and only if

$$\int_0^{j_{\alpha,2}} t^{-\beta} J_\alpha(t)dt \geq 0.$$

The proof of this theorem for $\alpha \in (-1, -1/2)$ is due to Askey and Steinig [5] and the case $\alpha \in (-1/2, 1/2)$ was proved by Makai [17].

Very recently Brown, Koumandos and Wang [8, 9] verified Gasper's conjecture for the cases when (α, β) lie on the lines $\alpha = \beta$ and $\alpha = -1/2$.

The objective of this note is to state a slight refinement of Conjecture 1 and to give numerical evidence for the truth of the new hypothesis.

2. The new conjecture

For any positive integer n, set

$$\Delta_n = \{(\alpha, \beta) \in D : \;\; S_n^{(\alpha,\beta)}(x) \geq 0 \;\text{ for }\; x \in [-1, 1]\}.$$

Then Gasper's conjecture may be formulated in the following equivalent form:

$$\bigcup_{n=1}^{\infty} \Delta_n = \Delta,$$

where Δ is defined by (3).

We state

Conjecture 2. *For any positive integer n, $\Delta_{n+1} \subset \Delta_n$.*

Denote by γ_n the boundary Δ_n, which passes through D:

$$\begin{aligned} \gamma_n \;\; = \;\; & \Big\{(\alpha_n, \beta_n) \in D \; : \; S_n^{(\alpha,\beta)}(x) \geq 0 \text{ for } x \in [-1, 1] \text{ and every } (\alpha, \beta) \text{ with} \\ & \alpha + \beta = \alpha_n + \beta_n, \;\; \beta \geq \beta_n \text{ and there exists } x \in [-1, 1] \; : \; S_n^{(\alpha,\beta)}(x) < 0 \\ & \text{for } (\alpha, \beta) \text{ with } \alpha + \beta = \alpha_n + \beta_n, \;\; \beta < \beta_n\}. \end{aligned}$$

The curve γ_n is well-defined bacause of Lemma 1.

An equivalent formulation of Conjecture 2 is that γ_{n+1} lies above γ_n for any positive integer n.

The latter conjecture implies Gasper's one because of the limit relation (4) and Theorem 1.

In the next section we give explicit expression for Δ_2 and Δ_3, or equivalently for γ_2 and γ_3. In section 3 an algorithm for tracing the curves γ_n is developed. Tables for the curves γ_n for $n = 4$ and 5 are given and the graphs of γ_n for $n = 2, 3, 4, 5$ are drawn. In Section 4 we discuss an idea how to prove Conjecture 2.

3. The cases $n = 2$ and $n = 3$

In what follows we suppose that $(\alpha, \beta) \in D$. First we consider the case $n = 2$. Straightforward calculations show that

$$4(\beta + 1)(\beta + 2)S_2^{(\alpha,\beta)}(x) = a_2x^2 + 2a_1x + a_0,$$

where

$$\begin{aligned} a_2 &= (\alpha+\beta+3)(\alpha+\beta+4), \\ a_1 &= 2(\alpha+2)(\alpha+\beta+3)+(\alpha+\beta+2)(\beta+2)-(\alpha+\beta+3)(\alpha+\beta+4) \\ &= (\alpha+1)(\alpha+\beta+4), \\ a_0 &= 2(\alpha+\beta+2)(\beta+2)+4(\alpha+1)(\alpha+2)+(\alpha+\beta+3)(\alpha+\beta+4)-4(\alpha+2)(\alpha+\beta+3) \\ &= \alpha^2+3\beta^2+3\alpha+7\beta+4. \end{aligned}$$

Obviously $S_2^{(\alpha,\beta)}(x)$ is convex and it attains its minimum value at $x_{min} = -a_1/a_2 = -(\alpha+1)/(\alpha+\beta+3)$ and $-1 < x_{min} < 0$. Hence $S_2^{(\alpha,\beta)}(x) \geq 0$ for $x \in [-1,1]$ if and only if it is nonnegative for any real x. Since its leading coefficient is positive, then $S_2^{(\alpha,\beta)}(x)$ is nonnegative if and only if its discriminant

$$(\alpha+1)^2(\alpha+\beta+4)^2-(\alpha+\beta+3)(\alpha+\beta+4)(\alpha^2+3\beta^2+3\alpha+7\beta+4)$$

is nonpositive. Thus,

$$\Delta_2 = \{(\alpha,\beta) \in D : \beta \geq \frac{-3\alpha-10+\sqrt{9\alpha^2+36\alpha+52}}{6}\}.$$

The case $n=3$ may be treated in a similar way because $S_n^{(\alpha,\beta)}(-1)=0$ for any odd n. Set $u=(x+1)/2$. Straightforward calculations show that

$$\bar{S}_3^{(\alpha,\beta)}(u) = \frac{S_3^{(\alpha,\beta)}(x)}{u} = b_2u^2-2b_1u+b_0$$

with

$$\begin{aligned} b_2 &= (\alpha+\beta+4)(\alpha+\beta+5)(\alpha+\beta+6)/(\beta+1)(\beta+2)(\beta+3), \\ b_1 &= (\alpha+\beta+4)(\alpha+\beta+6)/(\beta+1)(\beta+2), \\ b_0 &= 2(\alpha+\beta+4)/(\beta+1). \end{aligned}$$

We have to characterize the values of (α,β) in D, for which $\bar{S}_3^{(\alpha,\beta)}(u) \geq 0$ for each $u \in [0,1]$. Since $\bar{S}_3^{(\alpha,\beta)}(u)$ attains its minimum at $u_{min} = b_1/b_2 = (\beta+3)/(\alpha+\beta+5)$ and $u_{min} \in [0,1]$, then $\bar{S}_3^{(\alpha,\beta)}(u) \geq 0$ for $u \in [0,1]$ for those (α,β), for which the discriminant

$$\left(\frac{(\alpha+\beta+4)(\alpha+\beta+6)}{(\beta+1)(\beta+2)}\right)^2 - 2\frac{(\alpha+\beta+4)^2(\alpha+\beta+5)(\alpha+\beta+6)}{(\beta+1)^2(\beta+2)(\beta+3)}$$

of $\bar{S}_3^{(\alpha,\beta)}$ is nonnegative. Therefore

$$\Delta_3 = \left\{(\alpha,\beta) \in D \ : \ \beta \geq \frac{-\alpha-5+\sqrt{\alpha^2+6\alpha+17}}{2}\right\}.$$

4. An algorithm for finding Δ_n

The idea of the algorithm for tracing the curves γ_n is based on the following simple fact:

Lemma 2. *If $(\alpha_n,\beta_n) \in \gamma_n$, then there exists $\xi \in (-1,1)$, for which*

$$S_n^{(\alpha_n,\beta_n)}(\xi) = \frac{d}{dx}S_n^{(\alpha_n,\beta_n)}(\xi) = 0.$$

Proof. Assume that for some (α_n, β_n) the polynomial $S_n^{(\alpha_n,\beta_n)}(x)$ is positive at the points of local extrema in $(-1,1)$. Then continuous argument imply that there exists a neighbourhood U of (α_n, β_n), such that for every (α, β) in U and for every $x \in (-1,1)$ the polynomial $S_n^{(\alpha,\beta)}(x)$ is positive. The latter contradicts the definition of γ_n. The lemma is proved.

We need necessary conditions that a polynomial

$$p(x) = \sum_{\nu=0}^{n} a_\nu x^{n-\nu}$$

has a double root. Such conditions are known and are given in terms of the discriminant $D(p)$ of p,

$$D(P) = a_0^{2n-2} \prod_{1 \le i < j \le n} (x_i - x_j)^2,$$

where $x_1, \dots, x_n$ are the zeros of p.

Lemma 3. *The discriminant $D(p)$ of the polynomial p can be represented as a $(2n-1) \times (2n-1)$ determinant in the form*

$$\frac{a_0 D(p)}{(-1)^{n-1}} = \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n & & \\ na_0 & (n-1)a_1 & \cdots & a_{n-1} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & a_0 & a_1 & \cdots & a_{n-1} & a_n \\ & & na_0 & (n-1)a_1 & \cdots & a_{n-1} & \\ & & & na_0 & (n-1)a_1 & \cdots & a_{n-1} \end{vmatrix}.$$

Moreover, $D(p) = 0$ if and only if $p(x)$ has at least one zero with at least double multiplicity.

We refer to [18, Section 1.3.3] and the references therein for the proof of this lemma and for more information about discriminants.

Now the last two lemmas immediately yield:

Theorem 2. *Let $S_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} a_k(\alpha_n, \beta_n) x^{n-k}$. If $(\alpha_n, \beta_n) \in \gamma_n$, then*

$$D(\alpha_n, \beta_n) := D(S_n^{(\alpha_n,\beta_n)}) = 0.$$

The basic steps of the algorithm for constructing an approximation of the curve γ_n are:

1. Choose $k \in I\!N$.

2. Devide the interval $[-2, 1/2]$ into k subintervals by the mesh points $\alpha_n^{(i)} = -2 + 2.5i/k,\ i = 0, \dots, k$.

3. For any fixed $\alpha_n^{(i)}$ find all the solutions $\beta_{n,1}^{(i)}, \cdots, \beta_{n,p}^{(i)} \in (-1/2, 0)$ of the equation $D(\alpha_n^{(i)}, \beta) = 0$.

4. Find that $s,\ 1 \le s \le p$, for which

$$S_n^{(\alpha_n^{(i)},\beta_{n,s}^{(i)})}(x) \ge 0 \quad \text{for} \quad x \in [-1,1]$$

and

$$S_n^{(\alpha_n^{(i)},\beta_{n,s}^{(i)})}(\xi) = \frac{d}{dx} S_n^{(\alpha_n^{(i)},\beta_{n,s}^{(i)})}(\xi) = 0 \quad \text{for some} \quad \xi \in (-1,1).$$

5. Choose $\beta_n^{(i)} = \beta_{n,s}^{(i)}$.

6. Approximate the data $(\alpha_n^{(i)}, \beta_n^{(i)})$ by a smooth curve.

Table 1 contains the results of the algorithm for $n = 4$ and 5 for $k = 50$. The values of $\beta_4^{(i)}$ and $\beta_5^{(i)}$, which correspond to $\alpha_n^{(i)} = \alpha^{(i)} = -2 + 0.05i$, $i = 0, \dots, 50$, are given.

i	$\alpha^{(i)}$	$\beta_4^{(i)}$	$\beta_5^{(i)}$	i	$\alpha^{(i)}$	$\beta_4^{(i)}$	$\beta_5^{(i)}$
0	-2.00	0	0				
1	-1.95	-0.0124665	-0.0100482	26	-0.70	-0.29347	-0.271235
2	-1.90	-0.0248627	-0.020186	27	-0.65	-0.303304	-0.281463
3	-1.85	-0.0371837	-0.0304035	28	-0.60	-0.313026	-0.291642
4	-1.80	-0.0494251	-0.0406914	29	-0.55	-0.322637	-0.30177
5	-1.75	-0.0615829	-0.051041	30	-0.50	-0.332137	-0.311845
6	-1.70	-0.0736534	-0.0614439	31	-0.45	-0.341526	-0.321865
7	-1.65	-0.0856334	-0.0718924	32	-0.40	-0.350807	-0.331828
8	-1.60	-0.0975197	-0.0823791	33	-0.35	-0.359997	-0.341732
9	-1.55	-0.10931	-0.0928969	34	-0.30	-0.36904	-0.351576
10	-1.50	-0.121001	-0.103439	35	-0.25	-0.377995	-0.361359
11	-1.45	-0.132592	-0.114	36	-0.20	-0.386843	-0.371079
12	-1.40	-0.144079	-0.124573	37	-0.15	-0.395585	-0.380734
13	-1.35	-0.155462	-0.135153	38	-0.10	-0.404222	-0.390324
14	-1.30	-0.166739	-0.145734	39	-0.05	-0.412754	-0.399847
15	-1.25	-0.177909	-0.156312	40	0.00	-0.421183	-0.409303
16	-1.20	-0.18897	-0.166881	41	0.05	-0.429509	-0.418691
17	-1.15	-0.199922	-0.177438	42	0.10	-0.437734	-0.428009
18	-1.10	-0.210763	-0.187977	43	0.15	-0.445858	-0.437258
19	-1.05	-0.221493	-0.198469	44	0.20	-0.453883	-0.446436
20	-1.00	-0.232112	-0.208989	45	0.25	-0.46181	-0.455544
21	-0.95	-0.242619	-0.219454	46	0.30	-0.469638	-0.464579
22	-0.90	-0.253014	-0.229886	47	0.35	-0.477371	-0.473543
23	-0.85	-0.263296	-0.240284	48	0.40	-0.485008	-0.482435
24	-0.80	-0.273467	-0.250643	49	0.45	-0.49225	-0.491254
25	-0.75	-0.283524	-0.260961	50	0.50	-0.5	-0.5

TABLE 1. The curves γ_4 and γ_5

The graphs of the approximations of the curves γ_n for $n = 2, 3, 4$ and 5 (from below) are drawn in Figure 1 in the end of the paper.

5. An idea for proving conjecture 2

The graphs of the curves $\gamma_2, \gamma_3, \gamma_4$ and γ_5 show that Conjecture 2 holds for $n = 2, 3$ and 4. It is clear that Conjecture 2 would be proven if one proves that $S_n^{(\alpha,\beta)}$ is nonnegative on $[-1, 1]$ for any (α, β), for which $S_n^{(\alpha,\beta)}$ is nonnegative there. Another possible idea of proving Conjecture 2 is to show that for any $(\alpha_n, \beta_n) \in \gamma_n$ the inequality $S_{n+1}^{(\alpha_n,\beta_n)}(x) \geq 0$ fails for some $x \in [-1, 1]$. It turns out that for $n = 2, 3$ and 4 such x exist. Based on the graphs of $S_n^{(\alpha_n,\beta_n)}(x)$ and $S_{n+1}^{(\alpha_n,\beta_n)}(x)$ for various $(\alpha_n, \beta_n) \in \gamma_n$ we may state one more conjecture which implies the truth of Conjecture 2, and thus of Conjecture 1:

Conjecture 3. *Let* $(\alpha_n, \beta_n) \in \gamma_n$. *Then there exists a unique* $\xi_n \in (-1,1)$ *such that*

$$S_n^{(\alpha_n,\beta_n)}(\xi_n) = \frac{d}{dx} S_n^{(\alpha_n,\beta_n)}(\xi_n) = 0.$$

Moreover, there exist η_n' *and* η_n'' *with* $-1 < \xi_n < \eta_n' < \eta_n'' < 1$, *such that*

$$S_{n+1}^{(\alpha_n,\beta_n)}(x) < 0 \quad for \quad x \in (\eta_n', \eta_n'').$$

Finally, recall that Askey [3] conjectured that $\beta(\alpha)$, defined by (3) is a convex function, which is eqivalent to the statement that the curve γ is a convex one. It seems that every γ_n is a convex curve. If so, obviously γ also would be a convex one.

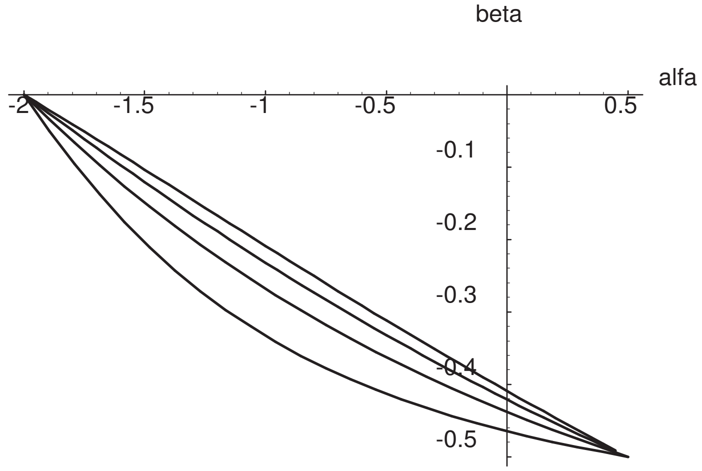

FIGURE 1. The curves $\gamma_2, \gamma_3, \gamma_4$ and γ_5, from below.

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