# CONVEXITY OF THE EXTREME ZEROS OF GEGENBAUER AND LAGUERRE POLYNOMIALS

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ABSTRACT. Let $C_n^\lambda(x)$, $n = 0, 1, \ldots$, $\lambda > -1/2$, be the ultraspherical (Gegenbauer) polynomials, orthogonal in $(-1, 1)$ with respect to the weight function $(1 - x^2)^{\lambda - 1/2}$. Denote by $x_{nk}(\lambda), k = 1, \ldots, n$, the zeros of $C_n^\lambda(x)$ enumerated in decreasing order. In this short note we prove that, for any $n \in I\!N$, the product $(\lambda + 1)^{3/2} x_{n1}(\lambda)$ is a convex function of $\lambda$ if $\lambda \geq 0$. The result is applied to obtain some inequalities for the largest zeros of $C_n^\lambda(x)$. If $x_{nk}(\alpha), k = 1, \ldots, n$, are the zeros of Laguerre polynomial $L_n^\alpha(x)$, also enumerated in decreasing order, we prove that $x_{n1}(\lambda)/(\alpha + 1)$ is a convex function of $\alpha$ for $\alpha > -1$.

## 1. INTRODUCTION

A classical result of Stieltjes [27] (see also [28, Theorem 6.21.1]) asserts that the positive zeros $x_{nk}(\lambda)$, $n \geq 2$, $k, 1 \leq k \leq [n/2]$ of $C_n^\lambda(x)$ are decreasing functions of $\lambda$. As pointed out by Ismail [18], this fact can be proved also by A. Markov's theorem [26] (see also [28, Theorem 6.12.1]) after a simple quadratic transformation. The fact that the positive zeros of $C_n^\lambda(x)$ decrease is intuitively clear from an interesting electrostatic interpretation of $x_{nk}(\lambda)$, $k = 1, \ldots, n$, as the positions of equilibrium of $n$ unit charges in $(-1, 1)$ in the field generated by two charges located at $-1$ and $1$ whose common value is $\lambda/2 + 1/4$ [28, pp. 140-142].

It has been of interest to do a deeper analysis of the behaviour of $x_{nk}(\lambda)$ as functions of $\lambda$. The problem of finding the extremal function $f_n(\lambda)$ which forces the products $f_n(\lambda)x_{nk}(\lambda)$, $k = 1, \ldots, [n/2]$, to increase was discussed in [23, 1, 16, 2, 11]. Elbert and Siafarikas [11] proved that $[\lambda + (2n^2 + 1)/(4n + 2)]^{1/2} x_{nk}(\lambda)$, $k = 1, \ldots, [n/2]$, are increasing functions of $\lambda$, for $\lambda > -1/2$, thus extending a result of Ahmed, Muldoon and Spigler [1] and proving a conjecture of Ismail, Letessier and Askey [19, 18]. Recently, the sharpness of the result of Elbert and Siafarikas was established in [5].

Convexity properties of these zeros have also been of interst. Elbert, Laforgia and Siafarikas [24, 9] conjectured, that the positive zeros of $C_n^\lambda(x)$ are convex functions of $\lambda$. This conjecture was supported by various facts. First of all, Elbert and Laforgia [8], established the asymptotic behaviour of $x_{nk}(\lambda)$,

$$(1.1) \qquad x_{nk}(\lambda) = h_{nk}\lambda^{-1/2} - \frac{h_{nk}}{8}\left(2n - 1 + 2h_{nk}^2\right)\lambda^{-3/2} + O(\lambda^{-5/2}), \quad \lambda \to \infty,$$

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where $h_{nk}$ denote the zeros of the Hermite polynomial $H_n(x)$ enumerated in decreasing order. Observe that (1.1) implies

$$\lim_{\lambda \to \infty} \lambda^{5/2} \frac{\partial^2 x_{nk}(\lambda)}{\partial \lambda^2} = \frac{3}{4} h_{nk}$$

and the latter yields the truth of the conjecture for sufficiently large values of $\lambda$. Moreover, Kokologiannaki and Siafarikas [22] proved that, for any $n \in I\!\!N$, the largest zero $x_{n1}(\lambda)$ is a convex function of $\lambda$ when $\lambda > n/\sqrt{3} + 1/2$. However, simple counterexamples were given in [4]. It is worth mentioning that the author of this note also strongly believed that all the positive zeros of $C_n^\lambda(x)$ were convex and even announced a "proof" of the conjecture for the largest zero $x_{n1}(\lambda)$ in [3]. On the other hand, the counterexample in [4] concerns exactly the largest zeros.

Before we state the main result, it seems appropriate to suggest a general problem of studying convexity/concavity properties of the positive zeros of $C_n^\lambda(x)$. Just as in the study of the monotonicity, it is reasonable to look for the "extremal" functions $g(\lambda)$, for which the products $g(\lambda)x_{nk}(\lambda)$, $n \geq 2$, $1 \leq k \leq [n/2]$, are convex or concave functions of $\lambda$. For various reasons, discussed in [2] and [11], we concentrate on functions $g(\lambda)$ of the form $g(\lambda) = (\lambda + c_n)^j$, where $c_n$ may depend on $n$ when we consider the convexity properties of the positive zeros of $C_n^\lambda(x)$ for $n$ fixed. Then the asymptotic formula (1.1) implies the following necessary conditions for the power $j$.

**Proposition 1.** *Let $n \in I\!\!N$. If $(\lambda + c_n)^j x_{nk}(\lambda)$ are convex (concave) functions of $\lambda$, for the whole range $\lambda > -1/2$, and for each $k$, $1 \leq k \leq [n/2]$, then $\lambda \notin (1/2, 3/2)$ ( $\lambda \in [1/2, 3/2]$ ).*

On the other hand, we already know from [4] that the largest zero $x_{n1}(\lambda)$ is not convex when $\lambda$ is small enough in comparison with $n$. Moreover, the asymptotic relation shows that it is natural to expect that there exists a sequence of constants $c_n$ for which $(\lambda + c_n)^{1/2} x_{nk}(\lambda), n \in I\!\!N, 1 \leq k \leq [n/2]$ are concave functions of $\lambda$. Because of that we are interested in the case when $j \geq 3/2$ and especially in the limit case $j = 3/2$.

**Theorem 1.** *For every $n \in I\!\!N$ the products*

$$(\lambda + 1)^{3/2} x_{n1}(\lambda)$$

*and*

$$\lambda^{3/2} x_{n1}(\lambda)$$

*are convex functions of $\lambda$ for $\lambda \geq 0$.*

Theorem 1 yields some new bounds for the largest zeros of the ultraspherical polynomials.

**Corollary 1.** *If $\lambda \in [0, 1]$ then*

$$(1.2) \qquad x_{n1}(\lambda) \leq (\lambda + 1)^{-3/2} \left( \lambda 2^{3/2} \cos \frac{\pi}{n+1} + (1 - \lambda) \cos \frac{\pi}{2n} \right).$$

*Moreover, for any $\lambda > 0$ we have*

$$(1.3) \ h_{n1} \lambda^{-1/2} - \frac{h_{n1}}{8} \left( 2n - 1 + 2h_{n1}^2 \right) \lambda^{-3/2} < x_{n1}(\lambda) < \left( \lambda + \frac{2n^2 + 1}{4n + 2} \right)^{-1/2} h_{n1}.$$

The first inequality is sharp when $n$ is fixed and $\lambda$ is close to 0 and to 1. However, even in these cases the right-hand side becomes greater than one when $\lambda$ is fixed and $n$ diverges.

Observe that the left-hand side inequality (1.3) shows that $x_{n1}(\lambda)$ is limited by the first two terms of its asymptotic expansion (1.1) not only for large values of $\lambda$ but for all $\lambda > 0$. The right-hand side inequality is a consequence of the above mentioned result of Elbert and Siafarikas and holds even for $\lambda > -1/2$.

In order to show how sharp (1.3) is for large values of $\lambda$, observe that for such values

$$(\lambda + c_n)^{-1/2} - \lambda^{-1/2} < -(c_n/2)\lambda^{-3/2},$$

where $c_n = (2n^2 + 1)/(4n + 2)$. On the other hand, (1.1) and an inequality about $x_{nk}(\lambda)$ which was announced recently by Elbert (see (2.10) in [6]), yields $h_{n1}^2 \leq 2n - 2$. Thus, for sufficiently large values of $\lambda$, (1.3) can be rewritten in the form

$$-\frac{3(2n-1)}{8} < \frac{\lambda^{3/2}}{h_{n1}}\left(x_{n1}(\lambda) - \frac{h_{n1}}{\sqrt{\lambda}}\right) < -\frac{2n^2 + 1}{4(2n + 1)}.$$

Moreover, the left inequality holds for any $\lambda > 0$.

The next result concerns the largest zeros $x_{n1}(\alpha)$ of the Laguerre polynomials $L_n^\alpha(x)$.

**Theorem 2.** *For every $n \in \mathbb{N}$ the quotient*

$$x_{n1}(\alpha)/(\alpha + 1)$$

*is a convex function of $\alpha$ for $\alpha > -1$.*

Ifantis and Siafarikas [17] proved that the function $x_{n1}(\alpha)/(\alpha + 1)$ decreases in the interval $(-1, +\infty)$. It is known that $nx_{nn}(\alpha) \to j_{\alpha,1}^2$, where $j_{\alpha,1}$ denotes the first positive zeros of the Bessel function $J_\alpha(x)$. Recently Elbert and Siafarikas [12] conjectured that $j_{\alpha,1}^2/(\alpha + 1)$ is a concave function of $\alpha$ for $\alpha > -2$, so we should expect that the smallest zero of $L_n^\alpha(x)$ satisfies a similar property, i.e., that $x_{nn}(\alpha)/(\alpha + 1)$ is concave at least for $\alpha > -1$. Numerical experiments support the latter conjecture.

## 2. Proofs of the Main Results

For the sake of completeness we shall provide a proof of Proposition 1. If $n \in \mathbb{N}$ is fixed and $c_n$ is any positive constant, then

$$
\begin{aligned}
((\lambda + c_n)^j x_{nk}(\lambda))'' &= j(j-1)(\lambda + c_n)^{j-2}x_{nk}(\lambda) \\
&\quad +2j(\lambda + c_n)^{j-1}x_{nk}'(\lambda) \\
&\quad +(\lambda + c_n)^j x_{nk}''(\lambda)
\end{aligned}
$$

Now (1.1) implies

$$((\lambda + c_n)^j x_{nk}(\lambda))'' = O\left((j^2 - 2j + 3/4)h_{nk}\,\lambda^{j-5/2}\right), \quad \lambda \to \infty,$$

which means that the products $(\lambda + c_n)^j x_{nk}(\lambda)$, $k = 1, \ldots, [n/2]$, are convex functions of $\lambda$, at least for sufficiently large $\lambda$, provided the binomial $j^2 - 2j + 3/4$ is nonnegative. The zeros of this binomial are $1/2$ and $3/2$.

By a parametric sequence of orthogonal polynomials we mean a class of orthogonal polynomial sequences $\{p_n(x;\tau)\}_{n=0}^\infty$ with $p_{-1}(x;\tau) \equiv 1$, where, in general, the

parameter $\tau$ is a vector. It indicates that the measures $d\psi(x;\tau)$ and the coefficients in the recurrence relation

$$(2.4) \quad x\, p_n(x;\tau) = \alpha_n(\tau)p_{n+1}(x;\tau) + \beta_n(\tau)p_n(x;\tau) + \gamma_n(\tau)p_{n-1}(x;\tau), \ n \geq 0,$$

vary with $\tau$. If $\tilde{p}_n(x;\tau)$, $n = 0, 1, \ldots$, are orthonormal then they satisfy a recurrence relation of the form

$$(2.5) \quad x\, \tilde{p}_n(x;\tau) = a_{n+1}(\tau)\tilde{p}_{n+1}(x;\tau) + b_n(\tau)\tilde{p}_n(x;\tau) + a_n(\tau)\tilde{p}_{n-1}(x;\tau),$$

with $a_n(\tau) > 0$, and we associate with $\tilde{p}_n(x;\tau)$ the $n \times n$ Jacobi matrix

$$J_n(\tau) = \begin{pmatrix} b_0(\tau) & a_1(\tau) & & & & \\ a_1(\tau) & b_1(\tau) & a_2(\tau) & & & \\ & a_2(\tau) & b_2(\tau) & a_2(\tau) & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1}(\tau) & b_{n-1}(\tau) \end{pmatrix}.$$

Denote by $\zeta_{n,k}(\tau)$, $k = 1, \ldots, n$, the zeros of $\tilde{p}_n(x;\tau)$ enumerated in decreasing order.

**Lemma 1.** a) *If the coefficients $b_k(\tau)$, $k = 0, \ldots, n-1$, and $a_k(\tau)$, $k = 1, \ldots, n-1$, are convex functions of $\tau$, for $\tau \in (p,q)$, then $\zeta_{n,1}(\tau)$ is a convex function of $\tau$, for $\tau \in (p,q)$.*
   b) *If $b_k(\tau)$, $k = 0, \ldots, n-1$, are concave, and $a_k(\tau)$, $k = 1, \ldots, n-1$, are convex functions of $\tau$, for $\tau \in (p,q)$, then $\zeta_{n,n}(\tau)$ is a concave function of $\tau$, for $\tau \in (p,q)$.*

*Proof.* It is well known that the zeros $\zeta_{n,k}(\tau)$ coincide with the eigenvalues of $J_n(\tau)$ and that an eigenvector, associated with $\zeta_{n,k}(\tau)$ is $(\tilde{p}_i(\zeta_{n,k}(\tau),\tau))_{i=0}^{n-1}$. When $a_k(\tau)$ and $b_k(\tau)$ are twice differentiable functions, since the zeros are distinct, the implicit function theorem yields that $\zeta_{n,k}(\tau)$ are also twice differentiable. Moreover, there is an explicit formula for the second derivative of the eigenvalues $\mu_k(\tau)$ of an Hermitian matrix $H(\tau)$, in the case when the eigenvalues are real and distinct. It has been rediscovered many times [25, 15] and reads as

$$(2.6) \quad \frac{d^2\mu_k(\tau)}{d\tau} = y_k^T \frac{d^2 H(\tau)}{d\tau^2} y_k + 2\sum_{j \neq k} \frac{1}{\mu_k - \mu_j}\left(y_k^T \frac{dH(\tau)}{d\tau} y_j\right)^2,$$

where $y_j$ form an orthogonal set of eigenvectors associated with $\mu_j$, $j = 1, \ldots, n$.

   Now we shall provide the proof of the lemma as stated, i.e., in the case when no smoothness is required. Its basic ingredients are the Perron–Frobenius theorem which states that the largest eigenvalues of positive irreducible matrices are increasing functions of each of its entries (see Theorems 8.4.4 and 8.4.5 on pp. 508 and 509 in [13]) and the Weyl theorem (see Theorem 4.3.1 on p. 181 in [13]), which states that, when $A$ and $B$ are Hermitian matrices, and the eigenvalues $\mu_k(A)$, $\mu_k(B)$ and $\mu_k(A+B)$ are arranged in decreasing order, then

$$(2.7) \quad \mu_k(A) + \mu_n(B) \leq \mu_k(A+B) \leq \mu_k(A) + \mu_1(B).$$

Since $\zeta_{n,k}(\tau)$ coincide with $\mu_k(J_n(\tau))$, statement a) of the lemma is equivalent to the convexity of $\mu_1(J_n(\tau))$. Without loss of generality we can assume that $b_k(\tau)$ are also positive. Then the right-hand side inequality (2.7) yields

$$(2.8) \quad \mu_1\left(\frac{J_n(\tau_1) + J_n(\tau_2)}{2}\right) \leq \frac{\mu_1(J_n(\tau_1)) + \mu_1(J_n(\tau_2))}{2}.$$

Now the convexity of the entries of $J_n(\tau)$ implies

$$J_n\left(\frac{\tau_1 + \tau_2}{2}\right) \le \frac{J_n(\tau_1) + J_n(\tau_2)}{2},$$

where the inequality is understood in the sense that inequalities hold for all the entries of the two matrices. Applying the Perron-Frobenius to the latter and combining the result with (2.8), we obtain

$$\mu_1\left(J_n\left(\frac{\tau_1 + \tau_2}{2}\right)\right) \le \frac{\mu_1(J_n(\tau_1)) + \mu_1(J_n(\tau_2))}{2}.$$

Statement a) is proved. For the proof of statement b) we need only a simple observation. Consider the matrix $\bar{J}_n$ whose diagonal entries are $-b_k(\tau)$ and the off-diagonal entries are again $a_k(\tau)$. Then the eigenvalues of the matrices $J_n$ and $\bar{J}_n$ are opposite to each other. $\square$

*Proof of Theorem* 1. It is well-known that the orthonormal ultraspherical polynomials are defined by the recurrence relation (2.5) with $b_k(\lambda) = 0$ and

$$a_k(\lambda) = \frac{1}{2}\left(\frac{k(k + 2\lambda - 1)}{(k + \lambda - 1)(k + \lambda)}\right)^{1/2}$$

Then the orthonormal polynomials, defined by

$$(2.9) \qquad x\, p_n(x; \lambda) = \frac{\sqrt{n+1}}{2}\sqrt{A_{n+1}(\lambda)}p_{n+1}(x; \lambda) + \frac{\sqrt{n}}{2}\sqrt{A_n(\lambda)}p_{n-1}(x; \lambda),$$

where

$$A_k(\lambda) = (\lambda + 1)^3 \frac{k + 2\lambda - 1}{(k + \lambda - 1)(k + \lambda)},$$

have zeros $(\lambda + 1)^{3/2}x_{n,k}(\lambda)$, $k = 1, \ldots, n$. Thus, by Lemma 1, all we need to prove is that $\sqrt{A_k(\lambda)}$, $k \ge 1$, are convex functions of $\lambda$ for $\lambda \ge 0$. On the other hand, for any sufficiently smooth function $f(\lambda)$ we have

$$\left(\sqrt{f(\lambda)}\right)'' = \frac{1}{4}\sqrt{f(\lambda)}\left\{2\frac{f''(\lambda)}{f(\lambda)} - \left(\frac{f'(\lambda)}{f(\lambda)}\right)^2\right\}.$$

Then we have to prove that

$$(2.10) \qquad 2\frac{A_k''(\lambda)}{A_k(\lambda)} - \left(\frac{A_k'(\lambda)}{A_k(\lambda)}\right)^2 \ge 0 \ \text{ for } \ \lambda \ge 0 \ \text{ and } \ k \ge 2,$$

because for $k = 1$ the quantity on the left-hand side vanishes. Lengthly but straight-forward calculations yield

$$2\frac{A_k''(\lambda)}{A_k(\lambda)} - \left(\frac{A_k'(\lambda)}{A_k(\lambda)}\right)^2 = \frac{(k-1)\left(r_5 + r_4\lambda + r_3\lambda^2 + r_2\lambda^3 + r_1\lambda^4\right)}{((\lambda + 1)(\lambda + k - 1)(\lambda + k)(2\lambda + k - 1))^2},$$

where $r_j = r_j(k)$ are polynomials of degree $j$ of the variable $k$. More precisely,

$$
\begin{aligned}
r_1(k) &= 23k - 31, \\
r_2(k) &= 8(8k^2 - 15k + 4), \\
r_3(k) &= 2(31k^3 - 74k^2 + 44k - 13), \\
r_4(k) &= 4(6k^4 - 17k^3 + 16k^2 - 11k + 4), \\
r_5(k) &= (k - 1)(3k^4 - 6k^3 + 5k^2 - 10k + 3).
\end{aligned}
$$

We have to prove that all these polynomials are positive for $k \geq 2$. This analysis is simple for $r_j$, when $j = 1, 2, 3$, and a little more complicated for $r_4$ and $r_5$. We shall perform it for $r_4$ because for $r_5$ it is similar. Since $r_4(2) > 0$, it suffices to show that $r_4'(t) > 0$ for $t \geq 2$. We have

$$r_4'(t) = 4 \left( 24(t-2)^3 + 93(t-2)^2 + 116(t-2) + 41 \right).$$

In order to prove the convexity of $\lambda^{3/2} x_{n1}(\lambda)$, we perform a similar procedure and in this case even the technicalities are less complicated. Now we have to prove (2.10) for

$$\tilde{A}_k(\lambda) = \lambda^3 \frac{k + 2\lambda - 1)}{(k + \lambda - 1)(k + \lambda)}.$$

We have

$$2 \frac{\tilde{A}_k''(\lambda)}{\tilde{A}_k(\lambda)} - \left( \frac{\tilde{A}_k'(\lambda)}{\tilde{A}_k(\lambda)} \right)^2 = \frac{\tilde{r}_6 + \tilde{r}_5 \lambda + \tilde{r}_4 \lambda^2 + \tilde{r}_3 \lambda^3 + \tilde{r}_2 \lambda^4}{\left( \lambda(\lambda + k - 1)(\lambda + k)(2\lambda + k - 1) \right)^2},$$

and all $\tilde{r}_j$ are nonnegative for $k \geq 1$. Indeed,

$$\begin{aligned}
\tilde{r}_2(k) &= 23k^2 - 18k + 7, \\
\tilde{r}_3(k) &= 4(k-1)(16k^2 - 7k + 1), \\
\tilde{r}_4(k) &= 2k(k-1)^2(31k - 5), \\
\tilde{r}_5(k) &= 24k^2(k-1)^3, \\
\tilde{r}_6(k) &= 3k^2(k-1)^4.
\end{aligned}$$

The proof of the theorem is complete.

**Lemma 2.** *Let $f \in C^2(a, \infty)$, where $a$ is a finite number or $-\infty$. If $f(t)$ is convex (concave) in $(a, \infty)$ and it has an asymptote $y(t) = kt + b$ at infinity, i.e. if $f(t) - kt - b \to 0$ as $t \to \infty$, then $f(t) \geq kt + b$ ($f(t) \leq kt + b$) for every $t \in (a, \infty)$. Moreover, if $f(t)$ is strictly convex (concave), then the latter inequalities are also strict.*

*Proof.* We consider only the case when $f(t)$ is convex. Set $g(t) := f(t) - kt - b$. Then the real axis is a horizontal asymptote of $g(t)$. Since $g(t)$ is also convex, then $g'(t)$ must be increasing in $(a, \infty)$. Assume that $g(x_0) < 0$ for some $x_0 > a$. Since $g(t) \to 0$ as $t \to \infty$, then there exists $t_1 \geq x_0$, such that $g'(t_1) > 0$.

On the other hand, for any $\varepsilon > 0$ there exists $A > t_1$, such that $|g(t)| < \varepsilon$ for any $t > A$. Choose $\varepsilon < g'(t_1)/4$ and two points $x_1$ and $x_2$ in $(A, \infty)$ whose distance is at least one. Then the mean values theorem implies that there is $t_2 \in (x_1, x_2)$, such that $g'(t_2) < g'(t_1)/2$. This contradicts the fact that $g'(t)$ is increasing in $(a, \infty)$. Hence $g(t) \geq 0$ for every $t > a$. $\qquad \square$

Now the statement of the corollary follows immediately. Inequality (1.2) is a consequence of the convexity of $(\lambda + 1)^{3/2} x_{n1}(\lambda)$ in $[0, 1]$. The left-hand side inequality (1.3) follows from the convexity of $\lambda^{3/2} x_{n1}(\lambda)$ in $(0, \infty)$ and the previous lemma.

*Proof of Theorem* 2. By the recurrence relation for the Laguerre polynomials [28, Eq. (5.1.10)], it is easy to see that the orthonormal polynomials, defined by the recurrence relation (2.5) with $b_k(\alpha) = (2k + \alpha + 1)/(\alpha + 1)$ and $a_k(\alpha) = $

$(k(k+\alpha))^{1/2}/(\alpha+1)$ have zeros $x_{nk}(\alpha)/(\alpha+1)$, $k=1,\ldots,n$. Then

$$b_k''(\alpha) = \frac{4k}{(\alpha+1)^3}$$

and

$$a_k''(\alpha) = \frac{k^{1/2}}{4(\alpha+1)^3(k+\alpha)^{3/2}}\left(3(\alpha+1)^3 + 12(k-1)(\alpha+1) + 8(k-1)^2\right).$$

## 3. Concluding Remarks and Open Problems

The statement of Lemma 1 for orthogonal polynomials with twice differentiable coefficients appears implicitly also in Ismail and Muldoon [20, Theorem 7], where the so-called birth and death process polynomials are considered. Though (2.6) may turn out to be a powerful tool for studying the convexity of zeros of orthogonal polynomials, it seems that the sum is rather intractable. Hence, the first term provides an essential piece of information which immediately implies the truth of Lemma 1 when the coefficients are smooth. The proof we provided requires no smoothness at all.

Formula (2.6) may be considered as a natural generalization of the Hellmann-Feynman theorem (see [14, 21]) for the derivatives of the zeros of orthogonal polynomials with respect to a parameter. It would be of interest to obtain generalizations of other results concerning monotonicity, such as Markov's theorem [26], [28, Theorem 6.12.1] which allows the monotonicity to be investigated trough the behaviour of the weight functions and the Stirm's type theorems [28, Section 6.3], [10], where the analysis can be done via the differential equation, satisfied by the orthogonal polynomials.

This necessity is motivated by the fact that very few results about convexity/concavity of zeros of orthogonal polynomials have been obtained. Though, as seen in the introduction, it is very risky to conjecture the behaviour of the higher derivatives of zeros of classical orthogonal polynomials, numerical evidence suggests that it might worth trying to prove the following facts.

1. Concerning the smallest zeros of $L_n^\alpha(x)$:

- $x_{nn}(\alpha)$ are convex function of $\alpha$ for all $n \in \mathbb{N}$, for $\alpha > -1$.
- $x_{nn}(\alpha)/(\alpha+1)$ are concave function of $\alpha$ for all $n \in \mathbb{N}$, for $\alpha > -1$.

The first of these conjectures is motivated by the convexity of $j_{\alpha,1}^2$, which was proved by Elbert and Laforgia [7] for $\alpha \geq 0$ and by Elbert and Siafarikas [12] for $-2 < \alpha < 0$, and the second by the above mentioned conjecture of Elbert and Siafarikas [12] that $j_{\alpha,1}^2/(\alpha+1)$ is concave.

2. Concerning the zeros of $C_n^\lambda(x)$:

- $(\lambda + c_n)^{3/2}x_{nk}(\lambda)$ are convex functions of $\lambda$,
- $(\lambda + c_n)^{1/2}x_{nk}(\lambda)$ are concave functions of $\lambda$,
- $(\lambda + c_n)x_{nk}^2(\lambda)$ are concave functions of $\lambda$,
- $(\lambda + c_n)^2x_{nk}^2(\lambda)$ are convex functions of $\lambda$,

where $c_n = (2n^2+1)/(4n+2)$ and all these properties hold for $\lambda \in (-1/2, \infty)$ and for all $n \in \mathbb{N}$ and $k=1,\ldots,[n/2]$.

The last two conjectures are somehow the best possible, because we can prove that, if $(\lambda + c_n)^j x_{nk}^2(\lambda)$ are convex (concave) functions of $\lambda$, for the whole range $\lambda > -1/2$, for every $n \in \mathbb{N}$ and each $k$, $1 \leq k \leq [n/2]$, then $\lambda \notin (1,2)$ ( $\lambda \in [1,2]$ ).

The second of the latter conjectures is also very challenging, because once we establish the concavity of $(\lambda + c_n)^{1/2} x_{nk}(\lambda)$, we shall obtain the monotonicity, i.e. the result of Elbert and Siafarikas [11] for granted. Indeed, the following simple lemma holds.

**Lemma 3.** *Let $f \in C^2(a, \infty)$, where $a$ is a finite number or $-\infty$. If $f(t)$ is convex (concave) in $(a, \infty)$ and it has a horizontal asymptote $y(t) = b$ at infinity, i.e. if $f(t) - b \to 0$ as $t \to \infty$, then $f'(t) \leq 0$ ($f(t) \geq 0$) for every $t \in (a, \infty)$. Moreover, if $f(t)$ is strictly convex (concave), then the latter inequalities are also strict.*

We omit the proof because it is similar to the proof of Lemma 2.

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