Connection Coefficients and Zeros of Orthogonal Polynomials

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Dedicated to Professor Theodore Chihara on his seventieth anniversary

Abstract

We discuss an old theorem of Obrechkoff and some of its applications. Some curious historical facts around this theorem are presented. We make an attempt to look at some known results on connection coefficients, zeros and Wronskians of orthogonal polynomials from the perspective of Obrechkoff's theorem. Necessary conditions in order the connection coefficients of two families of orthogonal polynomials to be positive are provided. Inequalities between the k-th zero of an orthogonal polynomial $p_n(x)$ and the largest (smallest) zero of another orthogonal polynomial $q_n(x)$ are given in terms of the signs of the connection coefficients of the families $\{p_n(x)\}$ and $\{q_n(x)\}$. An inequality between the largest zeros of the Jacobi polynomials $P_n^{(a,b)}(x)$ and $P_n^{(\alpha,\beta)}(x)$ is also established.

Key words: Connection coefficients, zeros of orthogonal polynomials, Descartes' rule of signs, Wronskians, inequalities for zeros.

1 Introduction

We discuss the relation between three topics on orthogonal polynomials. They are connection coefficients, zeros and Wronskians of sequences of such polynomials.

Let $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be two sequences of orthogonal polynomials with respect to different measures. Then for any positive integer n there is a unique

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sequence of numbers $a_{nk},\ k=0,\dots,n$ such that

$$q_n(x)=\sum_{k=0}^{n} a_{nk}p_k(x). \tag{1.1}$$

By a parametric sequence of orthogonal polynomials we mean a class of orthogonal polynomials sequences $\{p_n(x;\tau)\}_{n=0}^{\infty}$ where, in general, the parameter τ is a vector. It indicates that the measures $d\psi(x;\tau)$ and the coefficients in the recurrence relation

$$x\ p_n(x;\tau)=\alpha_n(\tau)p_{n+1}(x;\tau)+\beta_n(\tau)p_n(x;\tau)+\gamma_n(\tau)p_{n-1}(x;\tau) \tag{1.2}$$

vary with τ.

If $f_1(x),\dots,f_n(x)$ are sufficiently smooth functions the Wronskian $W(f_1,\dots,f_n;x)$ of $f_1(x),\dots,f_n(x)$ is defined by

$$W(f_1,\dots,f_n;x)=det\begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x)\\ f_1'(x) & f_2'(x) & \cdots & f_n'(x)\\ \vdots & \vdots & \vdots & \vdots\\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x)\end{pmatrix}.$$

Three important problems on the above topics topics are:
1. To characterize the pairs $\{p_n\},\ \{q_n\}$ of sequences of orthogonal polynomials for which the connection coefficients in (1.1) are all non-negative in terms of the coefficients in the recurrence relations satisfied by $\{p_n\}$ and $\{q_n\}$ or in terms of the measures associated with them.
2. To investigate the zeros of the polynomials as functions of the parameter τ through the behaviour of the measure $d\psi(x;\tau)$ or of the coefficients $\alpha_n(\tau),\ \beta_n(\tau)$ and $\gamma_n(\tau)$. Questions of particular interest are monotonicity and convexity/concavity of the zeros with respect to τ.
3. To find conditions under which the Wronskians of a finite subsequence of orthogonal polynomials do not change their signs on certain interval.

We employ a theorem of Obrechkoff to obtain a relation between the connection coefficients of two sequences of orthogonal polynomials and the behaviour of their zeros. This relation yields necessary conditions that the connection coefficients are non-negative. Conversely, some inequalities for zeros of classes of parametric orthogonal polynomials are derived in terms of the signs of their connection coefficients.

There are many important and interesting results on the three problems under discussion. Here are only the basic contributions:

It seems the first motivation of the importance of the first problem goes back to a work of Schoenberg [29] on positive definite functions. There Schoenberg conjectured that if $q_n(x) = C_n^{\lambda}(x)$ and $p_n(x) = C_n^{\mu}(x)$ are ultraspherical polynomials with $0 \le \mu < \lambda$, 2λ and 2μ integers, then the connection coefficients a_{nm} in (1.1) are non-negative. It turned out Gegenbauer [12] had a formula for the connection coefficients which implies the positivity of a_{nk} for $0 \le \mu < \lambda$ without the restriction that 2λ and 2μ are integers. Askey and Gasper [5] used Gegenbauer's result to prove the positivity of some ${}_3F_2$ polynomials $A_{n,k}(x)$ which played an important role in de Branges' proof [7] of the celebrated Bieberbach conjecture. Wilf [35] showed that $A_{k,n}(c)$ are nothing but the "connecting polynomials" of the Chebyshev polynomials of the second kind with shifted argument and the Chebyshev polynomials of the first kind,

$$U_n(c + (1-c)x) = \sum_{k=1}^{n} A_{k,n}(c) T_k(x).$$

As Wilf pointed out, the latter is a particular case of a result of Feldheim [10] who determined explicitly in terms of ${}_3F_2$ polynomials the corresponding connection coefficients for any families of Jacobi polynomials.

Let $p_n(x)$ and $q_n(x)$ be orthogonal with respect to the inner products $(f,g)_p := \int_{-\infty}^{\infty} f(x)g(x)\omega_p(x)dx$ and $(f,g)_q := \int_{-\infty}^{\infty} f(x)g(x)\omega_q(x)dx$, respectively. Askey [1] conjectured that if $\omega_p(x)$ and $\omega_p(x)$ are supported on $(0,\infty)$ and $\omega_q(x) = x^c\omega_p(x)$, with $c > 0$, $p_n(0) > 0$ and $q_n(0) > 0$ then the connection coefficients a_{nk} are positive. Wilson [36] proved that in this case $(p_i, p_j)_q \le 0$, $i \ne j$, $i, j = 0, \dots, n$ are sufficient conditions for a_{nk} to be non-negative. Trench [33] verified Wilson's conditions for weight functions of the above form thus proving Askey's conjecture. A short account of these results is given by Askey [3].

Let p_n and q_n, $n = 0, 1, \dots$ satisfy the recurrence relations

$$\begin{aligned} xp_n(x) &= \alpha_n(p)\ p_{n+1}(x) + \beta_n(p)\ p_n(x) + \gamma_n(p)\ p_{n-1}(x), \\ xq_n(x) &= \alpha_n(q)\ q_{n+1}(x) + \beta_n(q)\ q_n(x) + \gamma_n(q)\ q_{n-1}(x), \end{aligned}$$

where $\alpha_n(p)$, $\gamma_n(q)$, $\alpha_n(q)$ and $\gamma_n(q)$ are positive. Szwarc [32] proved that if all the vectors $(\alpha_k(p), \beta_k(p), \gamma_k(p), \alpha_k(p) + \gamma_k(p))$, $k = 0, 1 \dots, n$ majorize in the componentwise sense the vector $(\alpha_n(q), \beta_n(q), \gamma_n(q), \alpha_n(q) + \gamma_n(q))$ then the connection coefficients a_{nk} are non-negative.

Almost complete characterization of the pairs of Jacobi polynomials for which the connection coefficients are non-negative was given by Askey and Gasper [4]. We shall recall and apply their result in Section 4.

The problem of monotonicity of zeros of a parametric sequence of orthogonal polynomials has been of interest since Stieltjes' [30] and A.Markov's [20] fundamental contributions. Chapter 6 of Szegő's book [31] gives a complete account of older results. For the recent one we refer to the surveys of Ismail [13] and Muldoon [21]. Some of the important contributions on this topic will be of essential use in Section 3.

Doubtless the main contribution to the third problem is the comprehensive article of Karlin and Szegő [17] where variety of inequalities concerning Wronskians and Turánians associated with orthogonal polynomials are proved. Theorems 1 and 2 in [17] describe the number of zeros of Wronskians of a sequence of orthogonal polynomials $\{Q_n(x)\}$, where

$$Q_n(x) = k_n(-x) + \cdots, \quad k_n > 0.$$

According to Theorem 1, if l is even, then the Wronskians

$$W(n,l,x) := W(Q_n(x), \ldots, Q_{n+l-1}(x))$$

do not change their sign on the real line and, more precisely, $(-1)^{l/2}W(n,l,x) > 0$. Theorem 2 states that for odd l the polynomial $W(n,l,x),\ n = 0,1,\ldots$ has exactly n simple zeros and the zeros of two successive Wronskians $W(n,l,x)$ and $W(n+1,l,x)$ strictly interlace.

2 Obrechkoff's theorem

In this section we formulate Obrechkoff's theorem as well as some other results concerning sequences of functions which obey Descartes' rule of signs.

Definiton 2.1 *The finite sequence of functions $f_1, \ldots, f_n$ obeys Descartes' rule in the interval (a,b) if the number of zeros in (a,b) of any real linear combination*

$$\alpha_1 f_1(x) + \ldots + \alpha_n f_n(x)$$

does not exceed the number of sign changes in the sequence $\alpha_1, \ldots, \alpha_n$.

Theorem A (Obrechkoff [22]) *Let the sequence of polynomials $p_n(x)$, $n = -1,0,1,2,\ldots$ be defined recursively by $p_{-1}(x) = 0, p_0(x) = 1$,*

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x) \quad n \geq 0, \tag{2.3}$$

where $\alpha_n, \beta_n, \gamma_n \in I\!R,\ \ \alpha_n, \gamma_n > 0$. If $\zeta_n(p)$ denotes the largest zero of $p_n(x)$, then the sequence of polynomials $p_0, \ldots, p_n$ obeys Descartes' rule of signs in $(\zeta_n(p), \infty)$.

Having in mind Favard's theorem [9], [8, Theorem 4.4] we see that the requirements of the theorem are equivalent to the requirement that $\{p_n\}$ is a sequence of orthogonal polynomials. It is essential that the polynomials are normalized in such a way that their leading coefficients are positive.

Since it seems Theorem A is almost unknown it might worth providing some piece of history. Laguerre [18] proved Theorem A for the particular case when $p_n(x)$ are the Legendre polynomials $P_n(x)$. For the first time Theorem A, as stated above appears in the early paper of Obrechkoff [22]. After that he mentioned it many times [23,24]. M. Marden [19] gave a proof of Obrechkoff's theorem for the classical orthogonal polynomials. Laguerre's and Marden's proofs use the second order differential equation satisfied by the corresponding orthogonal polynomials. A well-known result of Bochner[6] (see also [8, p. 150]) states that the only orthogonal polynomials, associated with a real measure, which satisfy a second order differential equation are the classical one, so that Laguerre's and Marden's methods can not be extended to prove Obrechkoff's result. Schoenberg [28] found a new proof of Theorem A. The beauty of Obrechkoff's proof is in the ingenuous arguments he uses. Schoenberg's approach is based on powerful results. On the first stage he proves that $W(n,l,x) \geq 0$ for $x \geq \zeta_n(p)$ provided the leading coefficients of $p_k(x)$ are positive. Then he uses a result of Fekete [11] [16, Theorem 3.1] on totally positive matrices in order to prove the following result:

Theorem B *Let $p_n(x)$, $n = 0, 1, \ldots$ be a sequence of orthogonal polynomials with positive leading coefficients. Then for any sequence $\nu_1, \ldots, \nu_k$ of integers with $0 \leq \nu_1 < \ldots < \nu_k$ the inequalities*

$$W(p_{\nu_1}(x), p_{\nu_2}(x), \ldots, p_{\nu_k}(x)) \geq 0, \qquad x \geq \zeta_{\nu_k}(p) \tag{2.4}$$

hold.

Finally, a characterization of sequences of functions that obey Descartes' rule in terms of inequalities of the form (2.4) due to Pólya and Szegő [27, Part V, Problems 87, 90] is employed by Schoenberg.

3 Connection coefficients and zeros of orthogonal polynomials

We begin this section with an immediate consequence of Theorem A. However, it provides an interesting relation between connection coefficients of orthogonal polynomials and the behaviour of their largest zeros. This relation itself has various applications.

In what follows the number of sign changes (or equivalently the number of variations) in a finite sequence $a_1, a_2, \ldots, a_n$ will be denoted by $V(a_1, a_2, \ldots, a_n)$. The number of zeros (counting multiplicities) of a function f in the interval (a, b) will be denoted by $Z(f; (a, b))$.

Theorem 3.1 *Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be two sequences of orthogonal polynomials with positive leading coefficients. If $q_n(x)$ has the representation (1.1) in terms of a linear combination of $p_0(x), p_1(x), \ldots, p_n(x)$, then*

$$Z(q_n; (\zeta_n(p), \infty)) \leq V(a_{n0}, a_{n1}, \ldots, a_{nn}). \tag{3.5}$$

Theorem 3.2 *Let p_n, $n = 0, 1, \ldots$ and q_n, $n = 0, 1, \ldots$ be two sequences of orthogonal polynomials with positive leading coefficients and q_n has the representation (1.1) in terms of a linear combination of $p_0, p_1, \ldots, p_n$. If the coefficients $a_{nn}, \ldots, a_{n0}$ are all non-negative then there exists $k, 0 \leq k \leq n$ for which at least one of the inequalities*

$$\alpha_k(p) > \alpha_k(q), \quad \beta_k(p) > \beta_k(q), \quad \gamma_k(p) > \gamma_k(q) \tag{3.6}$$

holds.

Proof. Note first that the connection coefficients of two sequences of polynomials do not depend on shifts of the argument. The shift β of the argument changes only the coefficients $\beta_k(p)$ and $\beta_k(q)$ in the recurrence relations to $\beta_k(p) - \beta$ and $\beta_k(q) - \beta$, respectively. Thus without loss of generality we can assume that all the coefficients in recurrence relations are positive. For our purpose we do not need to change the denotations. Suppose the requirements in the theorem are satisfied but none of the inequalities (3.6) holds. Then the tridiagonal $n \times n$ matrix $A(q)$ with elements $a_{ij}(q) = \delta_{i,j+1}\gamma_i(q) + \delta_{i,j}\beta_i(q) + \delta_{i,j-1}\alpha_i(q)$ majorizes in the componentwise sense the corresponding matrix $A(p)$. The eigenvalues of the matrices $A(q)$ and $A(p)$ are the zeros of q_n and p_n, respectively. By the Perron-Frobenius theorem the largest zero $\zeta_n(q)$ of q_n is greater than the largest zero $\zeta_n(p)$ of p_n. Now Theorem 3.1 implies that there must be at least one sign change in the sequence of connection coefficients $a_{nn}, \ldots, a_{n0}$, a contradiction.

The following two immediate consequences of Theorem 3.2 concern the cases when the polynomials are orthonormalized or symmetric.

Corollary 3.1 *Let $\{p_n\}$ and $\{q_n,\}$ be two sequences of orthonormal polynomials with positive leading coefficients defined by*

$$xp_n(x) = \alpha_n(p)p_{n+1}(x) + \beta_n(p)p_n(x) + \alpha_{n-1}(p)p_{n-1}(x), \tag{3.7}$$
$$xq_n(x) = \alpha_n(q)q_{n+1}(x) + \beta_n(q)q_n(x) + \alpha_{n-1}(q)q_{n-1}(x). \tag{3.8}$$

Let $q_n(x)$ is represented in the form (1.1) and the coefficients $a_{nn}, \ldots, a_{n0}$ are *non-negative, then there exists $k, 0 \leq k \leq n$ for which at least one of the inequalities*

$$\alpha_k(p) > \alpha_k(q), \quad \beta_k(p) > \beta_k(q) \tag{3.9}$$

holds.

Corollary 3.2 *Let $P_n,\ n = 0, 1, \ldots$ and $Q_n,\ n = 0, 1, \ldots$ be two sequences of symmetric orthogonal polynomials with positive leading coefficients defined by*

$$xP_n(x) = \alpha_n(P)P_{n+1}(x) + \gamma_n(P)P_{n-1}(x)$$
$$xQ_n(x) = \alpha_n(Q)Q_{n+1}(x) + \gamma_n(Q)Q_{n-1}(x).$$

If the coefficients $a_{nn}, \ldots, a_{n0}$ in the representation of Q_n in terms of linear combination of $P_0, P_1, \ldots, P_n$ are all non-negative then there exists $k, 0 \leq k \leq n$ for which at least one of the inequalities

$$\alpha_k(P) > \alpha_k(P), \quad \gamma_k(P) > \gamma_k(Q) \tag{3.10}$$

holds.

On using Obrechkoff's result and the Hellmann-Feynman theorem (see [15]) we can prove the following result.

Theorem 3.3 *Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be two sequences of orthonormal polynomials satisfying the recurrence relations (3.7) and (3.8). If the coefficients $a_{nn}, \ldots, a_{n0}$ are all non-negative then the Jacobi matrix $A(q) - A(p)$, where $A(q)$ and $A(p)$ are defined as above, can not be positive definite. Therefore at least one principal minor of $A(q) - A(p)$ is negative.*

Recall now that a sequence $\{c_k\}_1^n$ is a finite chain sequence if and only if there exists a parametric sequence $\{g_k\}_0^n$ such that

$$c_k = g_k(1 - g_{k-1}), \quad k = 1, \ldots, n, \quad 0 \leq g_0 < 1, \quad 0 < g_k < 1, \quad k = 1, \ldots, n.$$

Corollary 3.3 *Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be two sequences of orthonormal polynomials with positive leading coefficients which satisfy the recurrence relations (3.7) and (3.8). If $\beta_k(q) - \beta_k(p) \geq 0$ and*

$$c_k(q,p) := \frac{(\alpha_k(q) - \alpha_k(p))^2}{(\beta_k(q) - \beta_k(p))(\beta_k(q) - \beta_k(p))}, k = 1, \ldots, n$$

is a chain sequence then at least one of the connection coefficients $a_{nn}, \ldots, a_{n0}$ in the representation (1.1) is negative. In particular, if $\beta_k(q) - \beta_k(p) \geq 0$ and $c_k(q,p) \leq (4\cos^2(\pi/(n+1)))^{-1}$ then at least one of the connection coefficients $a_{nn}, \ldots, a_{n0}$ is negative.

The proof of Corollary is based on Theorem A and results of Wall and Wetzel [34] and of Ismail and Li [14].

In order to simplify the formulation of the next results we shall say that a function f has at least one point of decrease in (a, b) in there exist $\xi \in (a, b)$ and $\epsilon > 0$ such that $\xi - \epsilon, \xi + \epsilon \in (a, b)$ and $f(\xi - \epsilon) > f(\xi - \epsilon)$.

Theorem 3.4 *Let $\{p_n(x;\tau)\}$ have positive leading coefficients and be orthogonal with respect to the measure $d\mu(x;\tau) = \omega(x;\tau)d\mu(x)$ on the interval (a, b) for any $\tau \in (c, d)$, where $\omega(x;\tau)$ is positive and has a continuous derivative with respect to τ for $x \in (a, b)$ and $\tau \in (c, d)$. Assume that the moments*

$$\int_a^b x^j \omega(x;\tau) d\mu(x), \quad j = 0, \ldots, 2n-1$$

converge uniformly for τ in every compact subinterval of (c, d). If the coefficients $a_{nk}(\tau_1, \tau_2)$ $k = 0, \ldots, n$ in the representation

$$p_n(x;\tau_1) = \sum_{k=0}^{n} a_{nk}(\tau_1, \tau_2) p_n(x;\tau_2)$$

are non-negative then

$$\frac{\partial \ln \omega(x;\tau)}{\partial \tau}$$

considered as a function of x has at least one point of decrease in (a, b).

For the proof we need to employ Theorem A and Markov's theorem [20], [31, Theorem 6.12.1]

4 Inequalities for zeros of orthogonal polynomials

In what follows we denote by $x_{n,k}(p)$ the zeros of $p_n(x)$ and suppose that they are arranged in decreasing order. Obrechkoff's theorem suggests the following straightforward estimates of some zeros of $q_n(x)$ in terms of the largest zeros of $p_n(x)$ provided the signs of the connection are known.

Theorem 4.1 *If $q_n(x)$ is represented by (1.1) and $V(a_{nn}, \ldots, a_{n0}) = k - 1$, then*

$$x_{n,k}(q) \leq x_{n,1}(p). \tag{4.11}$$

The connection coefficients of the classical orthogonal polynomials are known in an explicit form [2, Lecture 7]. They are given in terms of Pochhammer symbols and their signs can be easily determined.

For the ultraspherical polynomials Gegenbauer's result implies that

$$C_n^{(\mu)}(x) = \sum_{k=0}^{[n/2]} a_{nk}(\mu,\lambda) C_{n-2k}^{(\lambda)}(x),$$

where *sign* $a_{nk}(\mu,\lambda) =$ *sign* $(\mu-\lambda)_{(n-k)/2}$. This immediately yields

$$x_{n,k}(\mu) < x_{n,1}(\lambda) \quad \text{for} \quad \lambda - k + 1 < \mu < \lambda - k + 2.$$

Since the positive zeros of the Gegenbauer polynomials are decreasing functions of the parameter, we can conclude that

Corollary 4.1 *If $x_{n,k}(\lambda)$ denotes the k-th zero, in decreasing order of $C_n^{(\lambda)}(x)$, then*

$$x_{n,k}(\lambda) < x_{n,1}(\lambda + k - 2) \quad \text{for} \quad k = 1, \ldots, [n/2].$$

Laguerre polynomials require renormalization because with the common one the signs of their leading coefficients alternate. Let $l_n^{(\alpha)}(y) = L_n^{(\alpha)}(-y)$. Then the polynomials $l_n^{(\alpha)}(y)$ are orthogonal in $(-\infty, 0)$, their leading coefficients are positive and their zeros are $y_{n,n+1-k}(\alpha) = -x_{n,k}(\alpha)$, where $x_{n,k}(\alpha)$ denotes the k-th zero, in decreasing order, of $L_n^{(\alpha)}(x)$. Since

$$L_n^{(\beta)}(x) = \sum_{k=0}^{n} a_{nk}(\beta,\alpha) L_k^{(\lambda)}(x),$$

where *sign* $a_{nk}(\beta,\alpha) =$ *sign* $(\beta-\alpha)_{(n-k)/2}$, substituting $-y$ for x we conclude that

$$y_{n,j}(\beta) < y_{n,1}(\alpha) \quad \text{if} \quad \alpha - j < \beta < \alpha - j + 1.$$

These inequalities yield the following lower bounds for the zeros of $L_n^{(\beta)}(x)$ in terms of the smallest zeros of $L_n^{(\alpha)}(x)$.

Corollary 4.2 *If $x_{n,k}(\alpha)$ denotes the k-th zero, in decreasing order, of $L_n^{(\alpha)}(x)$, then for each k, $1 \le k \le n$,*

$$x_{n,k}(\beta) > x_{n,n}(\alpha) \quad \text{if} \quad \beta > \alpha - n + k - 1.$$

Similarly, denoting by $x_{n,k}(\alpha,\beta)$ the zeros of $P_n^{(\alpha,\beta)}(x)$ arranged in decreasing order and on using the explicit form of the connection coefficients of Jacobi polynomials

$$P_n^{(a,\beta)}(x) = \sum_{k=0}^{n} a_{nk}(a,\alpha,\beta) P_k^{(\alpha,\beta)}(x), \tag{4.12}$$

where *sign* $a_{nk}(a,\alpha,\beta) =$ *sign* $(a-\alpha)_{n-k}$, we can conclude that

$$x_{n,k}(a,\beta) < x_{n,1}(\alpha,\beta) \quad \text{if} \quad \alpha - k + 1 < a < \alpha - k + 2.$$

Then the fact that the zeros of Jacobi polynomials are decreasing functions of α implies:

Corollary 4.3 *If $x_{n,k}(\alpha,\beta)$ denotes the k-th zero, in decreasing order, of $P_n^{(\alpha,\beta)}(x)$, then for any k, $1 \le k \le n$,*

$$x_{n,k}(a,\beta) < x_{n,1}(\alpha+k,\beta) \quad \text{if} \quad a > \alpha - k + 1.$$

5 Monotonicity of largest zeros of Jacobi polynomials

In this section we are interested in the mutual location of the largest zeros of the zeros of two Jacobi polynomials. Given a pair (α,β), $\alpha,\beta > -1$, the problem is to find all (a,b), $a,b > -1$, for which the largest zero of $P_n^{(a,b)}(x)$ is smaller (greater) than the largest zero of $P_n^{(\alpha,\beta)}(x)$. In other words, the problem is to determine the pairs (a,b), such that the inequalities $x_{n,1}(a,b) < x_{n,1}(\alpha,\beta)$ (or $x_{n,1}(a,b) > x_{n,1}(\alpha,\beta)$) hold for every positive integer n. The well-known result of A. Markov [20], [31, Theorem 6.21.1] states that the zeros of $P_n^{(\alpha,\beta)}(x)$ are decreasing functions of α and increasing functions of β. Equivalently, the inequalities $x_{n,k}(a,b) < x_{n,k}(\alpha,\beta)$, $n = 2,3,\dots, k = 1,\dots,n$, hold for $a > \alpha$, $b < \beta$ and $x_{n,k}(a,b) > x_{n,k}(\alpha,\beta)$, $n = 2,3,\dots,$ $k = 1,\dots,n$, for $a < \alpha$, $b > \beta$. To the best of our knowledge nothing is know about the mutual location of $x_{n,k}(a,b)$ and $x_{n,k}(\alpha,\beta)$ when (a,b) belongs to the angles $\{a > \alpha,\ b > \beta\}$ and $\{a < \alpha,\ b < \beta\}$. Our result reads as follows:

Theorem 5.1 *If $\beta > \alpha$, then*

$$x_{n,1}(a,b) < x_{n,1}(\alpha,\beta), \qquad a > \alpha,\ b - \beta < a - \alpha. \tag{5.13}$$

If $\beta < \alpha$ and $\alpha + \beta > 0$, then

$$x_{n,1}(a,b) < x_{n,1}(\alpha,\beta), \qquad a > \alpha,\ b < \beta + \frac{\beta+1}{\alpha+1}(a-\alpha),\ (a,b) \notin \Delta, \tag{5.14}$$

where $\Delta = \Delta(\alpha,\beta)$ is the triangle with vertices at (α,β), $(2\alpha-\beta,\beta)$ and $(2\alpha+1, 2\beta+1)$.

Proof. Askey and Gasper [4] proved that the connection coefficients $a_{nk}(a,b;\alpha,\beta)$ in the representation (4.12) are non-negative in the following cases:
A) When $\beta > \alpha$ if (a,b) belongs to the region described in (5.13) except for a saw-tooth shaped set to the right of the vertical line through (α,β).
B) When $\beta > \alpha$ and $\alpha+\beta > 0$ if (a,b) belongs to a rather complicated region. It is the intersection of the three half-planes. The first is below the line which passes through (α,β) and $(-1,-1)$. The second is above the line with slope

-1 through $(\alpha\beta)$. The third half-plane is below the line with slope 1 through $(2\alpha, 2\beta)$.

Obrechkoff's theorem guarantees that $x_{n,1}(a, b) < x_{n,1}(\alpha, \beta)$ when (a, b) is in these regions. Markov's theorem allows us to extend this inequality to the regions described in (5.13) and (5.14) and this completes the proof.

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