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# Chebyshev-Laurent Polynomials and Weighted Approximation*

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Let $(a,b) \subset (0,\infty)$ and for any positive integer $n$, let $S_n$ be the Chebyshev space in $[a,b]$ defined by $S_n := span\{x^{-n/2+k}, k = 0,\ldots,n\}$. The unique (up to a constant factor) function $\tau_n \in S_n$, which satisfies the orthogonality relations $\int_a^b \tau_n(x)q(x)$ $(x(b-x)(x-a))^{-1/2}\,dx = 0$ for any $q \in S_{n-1}$ is said to be the orthogonal Chebyshev $S_n$-polynomials. This paper is an attempt to exibit some interesting properties of the orthogonal Chebyshev $S_n$-polynomials and to demonstrate their importance to the problem of approximation by $S_n$-polynomials. A simple proof of a Jackson-type theorem is given and the Lagrange interpolation problem by functions from $S_n$ is discussed. It is shown also that $\tau_n$ obeys an extremal property in $L_q$, $1 \le q \le \infty$. Natural analogues of some inequalities for algebraic polynomials, which we expect to hold for the $S_n$-polynomials, are conjectured.

## 1 Introduction

The distribution $d\psi(x)$ is said to be a strong distribution in $(a,b) \subset (0,\infty)$ if $\psi(x)$ is a real bounded nondecreasing function on $(a,b)$ with infinitely many points of increase there, and furthermore, all the moments

$$\mu_k = \int_a^b x^k d\psi(x), \quad k = 0, \pm 1, \pm 2, \ldots \qquad \text{exist.}$$

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For any given strong distribution in $(a,b)$ there exists a unique, up to a nonzero constant factor normalization, sequence of polynomials $\{B_n\}_0^\infty$ such that $B_n$ is a polynomial of precise degree $n$ and $B_n$ satisfies the relations

$$\int_a^b x^{-n+k} B_n(x) d\psi(x) = 0, \quad k = 0, \ldots, n-1. \tag{1.1}$$

These polynomials, may be called *orthogonal Laurent polynomials* or simply *orthogonal L-polynomials*. They were studied in details by Jones, Thron and Waadeland [9] in connection with the so-called strong Stieltjes moment problem.

Let $b_n(x) := x^{-n/2} B_n(x)$. Obviously $b_n \in S_n$, where

$$S_n := span\{x^{-n/2+k}, k = 0, \ldots, n\}. \tag{1.2}$$

In what follows the elements of $S_n$ will be called $S_n$-polynomials. Now the orthogonality relations (1.1) may be rewritten in the following more succint form:

$$\int_a^b b_n(x) q(x) x^{-1/2} d\psi(x) = 0 \quad \text{for any} \quad q \in S_{n-1}. \tag{1.3}$$

This latter orthogonality relation provides a natural relation between the spaces $S_n$ and the orthogonal $S_n$-polynomials $b_n$. The function $b_n$, which obeys the orthogonal property (1.3) with $d\psi(x) = ((b-x)(x-a))^{-1/2} dx$ will be called *orthogonal Chebyshev $S_n$-polynomial.*

An explicit representation for the orthogonal Chebyshev $S_n$-polynomials was obtained by Sri Ranga [15]. He proved that the $S_n$-polynomial $\tau_n$ defined by

$$\tau_n(x) := T_n\left(\frac{1}{\sqrt{b} - \sqrt{a}} \frac{x - \sqrt{ab}}{\sqrt{x}}\right), \tag{1.4}$$

where $T_n$ is the n-th Chebyshev polynomial of the first kind, satisfies the orthogonal properties (1.3). The orthogonal Chebyshev $S_n$-polynomial $\tau_n$ will be called simply *Chebyshev $S_n$-polynomial.*

Let $H_n = span\{g_0, g_1, \ldots, g_n\}$ be a Chebyshev space in $(a,b)$. Then any element of $H_n$ is called a $H_n$-polynomial. The problem of approximation by $H_n$-polynomials is stated as follows:

PROBLEM 1 *Given a continuous function $f$, find the quantity*

$$E_{H_n} := \inf_{g \in H_n} \|f - g\|_\infty,$$

*where $\|.\|_\infty$ is the uniform norm in $(a,b)$.*

If this problem has a unique solution $g^\star \in H_n$ such that $E_{H_n} = \|f - g^\star\|_\infty$, the $H_n$-polynomial $g^\star$ is said to be the best approximant of $f$. Another related problem is the problem of weighted approximation with varying weight, which can be stated in the following way:

Problem 2 *Given a continuous function $f$ and a weight function $W$ in $(a,b)$, find the quantity*

$$E_{H_n}(W) := \inf_{g \in H_n} \|f - W^n g\|_\infty.$$

Again, if the latter problem has a unique solution $g_W^\star \in H_n$ such that $E_{H_n}(W; f) = \|f - W^n g_W^\star\|_\infty$, then the function $W^n g_W^\star$ is said to be the best weighted approximant of $f$. The problem of weighted approximaton with varying weights has been of recent interest. A thorough study of this topic was given by Totik [18].

In this paper we consider Problem 1 for the Chebyshev space $H_n = S_n$, where $S_n$ is defined by (1.2). It is easy to see that in this case Problem 1 is equivalent to Problem 2 with weight $W(x) = 1/\sqrt{x}$ and Chebyshev space $H_n = I\!P_n$, the space of algebraic polynomials of degree not exceeding $n$. A very simple proof of a Jackson-type theorem is given in Section 3. The Lagrange interpolation problem by $S_n$-polynomials at the zeros of $\tau_n$ is also discussed there. In Section 4 we prove that the polynomials $\tau_n$ are solutions of an extremal problem in $L_p$. There we state also some conjectured inequalities concerning $S_n$-polynomials, which we find natutral analogs of the classical Bernstein, Markov and Szegö's inequalities for algebraic polynomials.

# 2 Basic Properties of the Chebyshev $S_n$-Polynomials

In this section we list some properties which follow immediately from the corresponding properties of the Chebyshev polynomial $T_n$. The books of Rivlin [14] and Szegö [17] contain comprehensive information about $T_n$.

Since $\tau_n(x) = T_n(u(x))$, where

$$u(x) = \frac{1}{\sqrt{b} - \sqrt{a}} \frac{x - \sqrt{ab}}{\sqrt{x}}, \tag{2.1}$$

we need some information about the transformation $x \to u(x)$. Note that

$$u'(x) = \frac{1}{2(\sqrt{b} - \sqrt{a})} \frac{x + \sqrt{ab}}{x^{3/2}} > 0, \quad \text{for} \quad x \in [a,b] \tag{2.2}$$

and

$$u''(x) = -\frac{1}{4(\sqrt{b} - \sqrt{a})} \frac{x + 3\sqrt{ab}}{x^{5/2}} < 0, \quad \text{for} \quad x \in [a,b]. \tag{2.3}$$

Inequality (2.2) and the equalities $u(a) = -1$ and $u(b) = 1$ show that $x \to u(x)$ is a one-to-one mapping between $[a,b]$ and $[-1,1]$. The well-known repesentations $T_n(u) = \cos(n \arccos u),\ u \in [-1,1]$, and $T_n(u) = \dfrac{(u + \sqrt{u^2-1})^n + (u + \sqrt{u^2-1})^n}{2}$ yield

$$\tau_n(x) = \cos\left(n \arccos(\frac{1}{\sqrt{b}-\sqrt{a}} \frac{x-\sqrt{ab}}{\sqrt{x}})\right), \quad x \in [a,b]$$

and

$$\begin{aligned} \tau_n(x) &= \frac{x^{-n/2}}{2(\sqrt{b}-\sqrt{a})^n} \Big( (x-\sqrt{ab}+\sqrt{(x-a)(x-b)})^n \\ &\quad + (x-\sqrt{ab}-\sqrt{(x-a)(x-b)})^n \Big). \end{aligned} \tag{2.4}$$

The latter representation shows that the polynomial

$$B_n(x) := (x-\sqrt{ab}+\sqrt{(x-a)(x-b)})^n + (x-\sqrt{ab}-\sqrt{(x-a)(x-b)})^n$$

obeys the orthogonal property (1.1) for $d\psi(x) = ((b-x)(x-a))^{-1/2}dx$.

The recurrence relation for the Chebyshev polynomials $T_n$ yields the following recurrence relation for the Chebyshev $S_n$-polynomials:

$$\begin{aligned} \tau_0(x) &= 1, \\ \tau_1(x) &= \frac{1}{\sqrt{b}-\sqrt{a}} \frac{x-\sqrt{ab}}{\sqrt{x}}, \\ \tau_{n+1}(x) &= \frac{2}{\sqrt{b}-\sqrt{a}} \frac{x-\sqrt{ab}}{\sqrt{x}} \tau_{n+1}(x) - \tau_{n+1}(x), \quad n = 1,2,\ldots. \end{aligned}$$

Recall that $|T_n(u)| \le 1$ for $u \in [-1,1]$, equality is attained only for $\eta_k = \cos(k\pi/n),\ k = 0,\ldots,n$, and $T_n(\eta_k) = (-1)^k$. Thus

$$|\tau_n(x)| \le 1, \quad \text{for} \quad x \in [a,b],$$

equality is attained only for

$$\tilde{\eta}_k = \frac{1}{4}\left( (\sqrt{b}-\sqrt{a})\cos(k\pi/n) + \sqrt{(\sqrt{b}-\sqrt{a})^2\cos^2(k\pi/n) + 4\sqrt{ab}} \right)^2, \quad k = 0,\ldots,n,$$

and $\tau_n(\tilde{\eta}_k) = (-1)^k$.

The zeros $\xi_k$ of $T_n$ are $\xi_k = \cos((2k-1)\pi/(2n)), k = 1,\ldots,n$. Hence the zeros $\tilde{\xi}_k, k = 1,\ldots,n$, of $\tau_n$ are

$$\tilde{\xi}_k = \frac{1}{4}\left( (\sqrt{b}-\sqrt{a})\cos\frac{(2k-1)\pi}{2n} + \sqrt{(\sqrt{b}-\sqrt{a})^2\cos^2\frac{(2k-1)\pi}{2n} + 4\sqrt{ab}} \right)^2.$$

Clearly $\tilde{\xi}_k$ are simple zeros of $\tau_n$ and $\tilde{\xi}_k \in (a,b),\ k = 1,\ldots,n$. Similarly $\tilde{\eta}_k \in [a,b],\ k = 0,\ldots,n$.

Performing the change of variables $u \to x$ in the differential identities $(1-u^2)(T_n'(u))^2 + n^2T_n^2 = n^2$ and $(1-u^2)T_n''(u) - uT_n'(u) + n^2T_n^2(u) = 0$, we conclude that $\tau_n$ is a solution of the differential equations

$$\frac{4(b-x)(x-a)x^2}{(x+\sqrt{ab})^2}(y')^2 + n^2y^2 = n^2 \tag{2.5}$$

and

$$4x^2(b-x)(x-a)(x + \sqrt{ab})y'' - 2x\Big(2x^3 + (4\sqrt{ab}-a-b)x^2$$
$$- 3\sqrt{ab}(a+b)x + 2(ab)^{3/2}\Big)y' + n^2(x+\sqrt{ab})^3 y = 0. \tag{2.6}$$

It is well-known that $|T_n'(u)| \le n^2$ for $u \in [-1,1]$ and equality is attained only for $u = \pm 1$. This fact together with (2.2) and (2.3) imply that

$$|\tau_n'(x)| \le \frac{\sqrt{b}+\sqrt{a}}{2a(\sqrt{b}-\sqrt{a})}n^2, \quad \text{for} \quad x \in [a,b] \tag{2.7}$$

and equality in (2.7) is attained only for $x = a$.

We conclude this section with an identity for the Chebyshev $S_n$-polynomials for the particular case when $ab = 1$ Observe that the method that Sri Ranga used to obtained the explicit expression (1.4) implies that when $ab = 1$ the corresponding orthogonal L-polynomials $B_n$ are self-inverse. Recall that a polynomial $p$ is called self-inverse if for any of its zeros $z_i$ its inverse $1/\bar{z}_i$ is also a zero of $p$. O'Hara and Rodriguez [13] proved that for every self-inversed polynomial of degree the equality $|np(z)/(zp'(z)) - 1| = 1$ holds for each z on the unit circle in the complex plane. This implies:

*If* $ab = 1$ *and the Chebyshev* $S_n$*-polynomial* $\tau_n$ *be defined by* (1.4), *then*

$$|n\tau_n(z) - 2z\tau_n'(z)| = |n\tau_n(z) + 2z\tau_n'(z)| \quad \textit{for each} \quad z,\ |z| = 1.$$

# 3 Approximation and interpolation by $S_n$-polynomials.

Consider the space $S_n$ defined by (1.2) and the distinct points $x_1, x_2, \ldots, x_{n+1}$, in the interval $[a,b]$, $0 < a < b < \infty$. Let the determinant $\Delta_n$ be defined by

$$\Delta_n := \begin{vmatrix} x_1^{-n/2} & x_1^{-n/2+1} & \cdots & x_1^{n/2} \\ x_2^{-n/2} & x_2^{-n/2+1} & \cdots & x_2^{n/2} \\ \vdots & \vdots & & \vdots \\ x_{n+1}^{-n/2} & x_{n+1}^{-n/2+1} & \cdots & x_{n+1}^{n/2} \end{vmatrix}.$$

Then,

$$\Delta_n = \prod_{i=1}^{n+1} x_i^{-n/2} \prod_{1\le i<j\le n+1} (x_j - x_i) \ne 0. \tag{3.1}$$

Therefore, since $x^{-n/2+k} \in C[a,b]$, $S_n$ is a Chebyshev system in $[a,b]$.

For any positive integer $n$, we define the space $\bar{S}_n$ by $\bar{S}_n := span\{1 \cup S_n\}$. Observe that $\bar{S}_n$ coincides with $S_n$ when $n$ is even. The modulus of continuity of a function $f$, which is continuous in $[a,b]$, is defined by $\omega(f;\delta) := \sup\{|f(x)-f(y)| : x,y \in [a,b], |x-y| \leq \delta\}$.

THEOREM 1 *Let $f \in C[a,b]$. Then for every $n \in I\!N$ there exists $s_n^\star \in \bar{S}_n$ such that*

$$\|f - s_n^\star\|_\infty \leq D\omega(f; 1/\sqrt{n}),$$

*where the constant $D$ depends on $a$ and $b$.*

Proof: Recall that (see [6]) the general Chebyshev polynomial $h_{H_n}$ associated with the Chebyshev space $H_n := span\{g_0, ..., g_n\}$ is the linear form

$$h_{H_n} := c\left(g_n - \sum_{i=0}^{n-1} c_i g_i\right),$$

where $c_i$ are chosen so that $\sum_0^{n-1} c_i g_i$ is the best uniform approximation on $[a,b]$ to $g_n$ from $span\{g_0, ... g_{n-1}\}$, and where $c$ is chosen such that $\|h_{H_n}\|_\infty = 1$ and $h_{H_n}(b) = 1$. This uniquely determines $h_{H_n}$. It is known that $h_{H_n}$ has exactly $n$ zeros $x_1 < ... < x_n$ in $[a,b]$. If $x_0 := a$ and $x_n := b$, then we define the mesh of $h_{H_n}$ by

$$M_{H_n} := \max_{1\leq i\leq n+1} |x_i - x_{i-1}|.$$

Using the properties of $\tau_n$ listed in Section 2 one can easily see that this is the general Chebyshev polynomials associated with $S_n$ and the mesh $M_{S_n}$ of $\tau_n$ is given by

$$M_{S_n} := \max_{1\leq i\leq n+1} |\tilde{\xi}_i - \tilde{\xi}_{i-1}|,$$

where $\tilde{\xi}_0 = a$ and $\tilde{\xi}_{n+1} = b$. Borwein and Staff [7, Lemma 1 a)] proved that if the above Chebyshev space $H_n$ satisfies: 1) $g_i \in C'[a,b]$ and 2) $H_n' := span\{g_0', ..., g_n'\}$ is also a Chebyshev space in $[a,b]$ then for any $f \in C[a,b]$ there exists $\bar{h}_n \in \bar{H}_n := span\{1, g_0, ..., g_n\}$, such that

$$\|f - \bar{h}_n\|_\infty \leq D\omega(f; \sqrt{M_{H_n}}),$$

where the constant $D$ depends an $a$ and $b$. Observe that all the requirements of the above statement are satisfied if $H_n = S_n$. Indeed, $x^{-n/2+k} \in C^1[a,b]$ and $S_n' := span\{(x^{-n/2+k})', k = 0, ..., n\}$ is a Chebyshev space in $[a,b]$. Futhermore, we have $\bar{H}_n = \bar{S}_n$. On applying Borwein and Staff's lemma we conclude that for any $f \in C[a,b]$ there exists $s_n^\star \in \bar{S}_n$ such that

$$\|f - s_n^\star\|_\infty \leq D\omega(f; \sqrt{M_{S_n}}),$$

where the constant $D$ depends on $a$ and $b$. Observe that

$$\tilde{\xi}_i - \tilde{\xi}_{i-1} = u^{-1}(\xi_i) - u^{-1}(\xi_{i-1}) = \frac{du^{-1}(\theta)}{d\xi}(\xi_i - \xi_{i-1}), \quad \theta \in (\xi_i, \xi_{i-1}),$$

where $u^{-1}$ is the inverse mapping of the mapping $u$ defined by (2.1). Since $u^{-1}$ has a bounded derivative in $[a,b]$ and $|\xi_i - \xi_{i-1}| \le const\ n^{-1}$, then $\sqrt{M_{S_n}} \le const\ 1/\sqrt{n}$. Now the statement of the theorem follows from the well-known property $\omega(f;\lambda\delta) \le (\lambda+1)\omega(f;\delta)$ of the modulus od continuity. □

Let

$$E_{\bar{S}_n}(f;[a,b]) := \inf_{s\in\bar{S}_n} \|f-z\|_\infty \tag{3.2}$$

be the best approximation of $f$ by $\bar{S}_n$-polynomials. Theorem 1 implies that for any $f \in C[a,b]$ we have $E_{\bar{S}_n}(f;[a,b]) \to 0$ as $n\to\infty$. In particular, $E_{\bar{S}_n}(f;[a,b]) \to 0$ as $n \to \infty$ when $n$ is even.

We now consider the Lagrange interpolation problem by $S_n$-polynomials: Given a continuous function $f(x)$ and distinct points $x_1, x_2, \ldots, x_{n+1}$ in $[a,b]$, determine a $S_n$-polynomial $\mathcal{L}_n(f;x)$ such that $\mathcal{L}_n(f;x_i) = f_i := f(x_i)$ for $i = 1, ..., n+1$.

This problem has a unique solution since $\Delta_n \ne 0$. Moreover, $\mathcal{L}_n(f;x)$ can be explicitly represented in the form

$$\mathcal{L}_n(x) := \mathcal{L}_n(f;x) := \sum_{k=1}^{n+1} f_k \ell_k(x). \tag{3.3}$$

Here

$$\ell_k(x) = \frac{\Omega(x)}{(x-x_k)\Omega'(x_k)}, \quad k = 1, \ldots, n+1, \tag{3.4}$$

where $\Omega(x) = x^{-n/2}\omega(x)$ and $\omega(x) = \prod_{i=1}^{n+1}(x-x_i)$, are the basic $S_n$-polynomials, for which $\ell_k(x_i) = \delta_{kj}$.

THEOREM 2 *Let $f(x)$ defined in $[a,b] \subset (0,\infty)$ and $x^{n/2}f(x) \in C^n[a,b]$. Suppose that $\dfrac{d^{n+1}(x^{n/2}f(x))}{dx^{n+1}}$ exists everywhere in $(a,b)$. Then*

$$f(x) - \mathcal{L}_n(x) = \frac{\Omega(x)}{(n+1)!}\left[\frac{d^{n+1}}{dx^{n+1}}(x^{n/2}f(x))\right]_{x=\theta},$$

*where $\min(x, x_1, \ldots, x_{n+1}) < \theta < \max(x, x_1, \ldots, x_{n+1})$. The point $\theta$ depends on $x, x_1, \ldots, x_{n+1}$ and $f$. Moreover, if the nodes of interpolation coincide with the zeros $\{\tilde{\xi}_i\}_1^{n+1}$ of $\tau_{n+1}$ then the error of interpolation*

$$R_n(f;x) = \|f(x) - \mathcal{L}_n(f;x)\|_\infty \tag{3.5}$$

*can be estimated in the following way:*

$$
\begin{aligned}
R_n(x) &\leq \frac{\|\Omega(x)\|_\infty}{(n+1)!} \left\| \frac{d^{n+1}}{dx^{n+1}} \left( x^{n/2} f(x) \right) \right\|_\infty \\
&\leq \frac{(\sqrt{b}-\sqrt{a})^{n+1}\sqrt{b}}{2^n (n+1)!} \left\| \frac{d^{n+1}}{dx^{n+1}} \left( x^{n/2} f(x) \right) \right\|_\infty \\
&:= \left\| \frac{d^{n+1}}{dx^{n+1}} \left( x^{n/2} f(x) \right) \right\|_\infty \tilde{R}_{S_n}.
\end{aligned}
$$

The proof of this result is analogous to the polynomial case.

Let us now compare the usual Lagrange interpolation by algebraic polynomials of degree not exceeding $n$, with the above interpolation by $S_n$-polynomials.

Let $\hat{\xi}_1 < \ldots < \hat{\xi}_{n+1}$ be the zeros of the Chebyshev polynomial $\hat{T}_{n+1}(x) = T_{n+1}\left(\frac{2x-(a+b)}{b-a}\right)$. The Lagrange interpolating polynomial of $f(x)$ at $\hat{\xi}_1, \ldots, \hat{\xi}_{n+1}$ is then given by

$$
L_n(x) := L_n(f;x) = \sum_{k=1}^{n+1} \hat{f}_k \hat{l}_k(x),
$$

where $\hat{l}_k(x) = \dfrac{\hat{\omega}(x)}{(x-\hat{\xi}_k)\hat{\omega}'(\hat{\xi}_k)}$, $k = 1, \ldots, n+1$ , $\hat{\omega}(x) = (x-\hat{\xi}_1)(x-\hat{\xi}_2)\ldots(x-\hat{\xi}_{n+1})$ and

$$
\begin{aligned}
r_n(f;x) &:= \|f - L_n(x)\|_\infty \\
&\leq \frac{\|\hat{\omega}(x)\|_\infty}{(n+1)!} \|f^{(n+1)}(x)\|_\infty \\
&\leq \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!} \|f^{(n+1)}(x)\|_{[a,b]} \\
&:= \|f^{(n+1)}\|_\infty \tilde{R}_n.
\end{aligned}
$$

It is interesting that

$$
\frac{\tilde{R}_{S_n}}{\tilde{R}_n} = \frac{2^{n+1}\sqrt{b}}{(\sqrt{b}+\sqrt{a})^{n+1}},
$$

i.e., the error estimate we have obtained for the interpolation by $S_n$-polynomials is better than the error estimate for the interpolation by algebraic polynomials for all sufficiently large $n$ if $\sqrt{a} + \sqrt{b} > 2$.

We shall compare also the error of interpolation with the best approximation by $S_n$-polynomials.

THEOREM 3 *Let $n$ be a positive even integer, $f \in C[a,b]$ and, let $R_n(f;x)$ and*

$E_{\bar{S}_n}(f;x)$ *be defined by* (3.5) and (3.2), respectively. Then the inequality

$$R_n(f;x) \le E_{\bar{S}_n}(f)(1 + \max_{a\le x\le b} \sum_{j=1}^{n+1} |\ell_j(x)|)$$

holds. Moreover, if the nodes of interpolation coincide with the zeros $\{\tilde{\xi}_i\}_{i=1}^{n+1}$ of $\tau_{n+1}$ then

$$R_n(f;x) \le \left(1 + \frac{b}{a}(1 + \frac{2}{\pi}\log n)\right) E_{\bar{S}_n}(f).$$

Proof: Since $n$ is a even integer, we have $\bar{S}_n = S_n$. Let $s_n^\star$ be the best approximant of $f$ from $S_n$. This means that

$$\|s_n^\star - f\|_\infty = E_{\bar{S}_n}(f).$$

But,

$$|f(x) - \mathcal{L}_n(f;x)| \le |f(x) - s_n^\star(x)| + |s_n^\star(x) - \mathcal{L}_n(f;x)|.$$

¿From the unicity of $\mathcal{L}_k(f;x)$, we have $s_n^\star(x) = \mathcal{L}_k(s_n^\star;x)$, and then $s_n^\star(x) - \mathcal{L}_n(f;x) = \mathcal{L}_n(s_n^\star - f;x)$. Thus,

$$|f(x) - \mathcal{L}_n(f;x)| \le E_{\bar{S}_n}(f) + |\mathcal{L}_n(s_n^\star - f;x)|. \tag{3.6}$$

Now,

$$\begin{aligned} |\mathcal{L}_n(s_n^\star - f)| &= |\sum_{j=1}^{n+1} \ell_j(x)(s_n^\star(\tilde{\xi}_j) - f(\tilde{\xi}_j))| \\ &\le \max_{a\le x\le b} |s_n^\star(x) - f(x)| \max_{a\le x\le b} \sum_{j=1}^{n+1} |\ell_j(x)| \\ &= E_{\bar{S}_n} \max_{a\le x\le b} \sum_{j=1}^{n+1} |\ell_j(x)|. \end{aligned} \tag{3.7}$$

The first statement of the theorem follows immediately from (3.6) and (3.7).

In order to prove the second one, we shall estimate the size of the constant $\max_{a\le x\le b} \sum_{j=1}^{n+1} |\ell_j(x)|$.

Consider the interpolating polynomials

$$\mathcal{L}_n(f;x) = \sum_{j=1}^{n+1} \ell_j(x) f(\tilde{\xi}_j) \in S_n \quad \text{and} \quad L_n(f;x) = \sum_{j=1}^{n+1} l_j(x) f(\xi_j) \in \mathbb{P}_n$$

over the zeros of $\tau_{n+1}(x)$ and $T_{n+1}(x)$, respectively. Then we have

$$\ell_j(x) = \frac{\tau_{n+1}(x)}{(x - \tilde{\xi}_j)\tau'_{n+1}(\tilde{\xi}_j)} \quad \text{and} \quad l_j(x) = \frac{T_{n+1}(x)}{(x - \xi_j)T'_{n+1}(\xi_j)}\ ,\ j = 1, \ldots, n+1.$$

Equations (1.4) and (2.1) imply

$$|\ell_j(x)| = \frac{|T_{n+1}(u(x))|}{|T'_{n+1}(\xi_j)||x-\xi_j|}\left(\frac{2(\sqrt{b}-\sqrt{a})x\sqrt{x}}{x+\sqrt{ab}}\right)\frac{|x-\xi_j|}{|x-\tilde{\xi}_j|}.$$

On the other hand, the inequality

$$\left|\frac{x-\xi_j}{x-\tilde{\xi}_j}\right| = \frac{1}{\sqrt{b}-\sqrt{a}}\left(\frac{\sqrt{\tilde{\xi}_j x}+\sqrt{ab}}{\sqrt{\tilde{\xi}_j x}(\sqrt{x}+\sqrt{\tilde{\xi}_j})}\right) \le \frac{1}{\sqrt{b}-\sqrt{a}}\left(\frac{\sqrt{a}+\sqrt{b}}{2a}\right)$$

holds because the function on the left-hand side of this inequality is a decreasing one in $[a,b]$.

It is easy to see that $\left(\frac{2(\sqrt{b}-\sqrt{a})x\sqrt{x}}{x+\sqrt{ab}}\right)$ is a increasing function on $[a,b]$. This fact immediately yields

$$|\ell_j(x)| \le \frac{b}{a}\ |l_j(u(x))|\ ,\ j=1,\ldots,n+1$$

and then

$$\max_{a\le x\le b}\sum_{j=1}^{n+1}|\ell_j(x)| \le \frac{b}{a}\ \max_{-1\le u\le 1}\sum_{j=1}^{n+1}|l_j(u(x))|.$$

Now we employ the well-known estimate

$$\max_{-1\le u\le 1}\sum_{j=1}^{n+1}|\ell_j(x)| \le 1+\frac{2}{\pi}\log n$$

for the Lebesgue's function in order to complete the proof. □

# 4 Extremal Properties of the Chebyshev $S_n$-polynomials

Consider the following extremal problem:

Problem 3 *Let $1\le q\le\infty$ , $n\in I\!N$ be given and*

$$\|f\|_{q,n} := \int_a^b |x^{-n}f(x)|^q\ ((b-x)(x-a))^{-1/2}dx.$$

*Find the polynomials $r^\star_{2n-1}\in I\!P_{2n-1}$ for which*

$$\|x^{2n}-r^\star_{2n-1}\|_{q,n} = \inf_{r_{2n-1}\in I\!P_{2n-1}}\|x^{2n}-r_{2n-1}\|_q$$

THEOREM 4 *For any $q$, $1 \leq q \leq \infty$ and $n \in I\!N$* PROBLEM 3 *has a unique solution $r^\star_{2n-1}$ and it is*

$$r^\star_{2n-1}(x) = x^{2n} - 2^{-2n+1}(\sqrt{b}-\sqrt{a})^{2n}x^n\tau_{2n}(x).$$

*The extremal polynomial $x^{2n} - r^\star_{2n-1}(x)$ may be written also in the following equivalent form:*

$$\begin{aligned} x^{2n} - r^\star_{2n-1}(x) &= \frac{(\sqrt{b}-\sqrt{a})^{2n}}{2^{2n-1}}\Big((x-\sqrt{ab}+\sqrt{(x-a)(x-b)})^{2n} \\ &+ (x-\sqrt{ab}-\sqrt{(x-a)(x-b)})^{2n}\Big). \end{aligned} \tag{4.1}$$

Proof: Given a polynomial $Q \in I\!P_m$, such that $Q(x) > 0$ for $x \in [a,b]$ and an integer $\nu \geq 2m$, let $r^\star_{\nu-1} \in I\!P_{\nu-1}$ be the unique solution of the extremal problem

$$\|(x^\nu - r^\star_{\nu-1}(x))/\sqrt{Q(x)}\|_\infty = \inf_{r_{\nu-1}\in I\!P_{\nu-1}} \|(x^\nu - r_{\nu-1}(x))/\sqrt{Q(x)}\|_\infty.$$

Bernstein in [2, 3] proved that for any $q$, $1 \leq q \leq \infty$, $r^\star_{\nu-1}$ is also the unique solution of the extremal problem

$$\int_a^b |(x^\nu - r^\star_{\nu-1}(x))/\sqrt{Q(x)}|^q\ ((b-x)(x-a))^{-1/2}dx =$$

$$\inf_{r_{\nu-1}\in I\!P_{\nu-1}} \int_a^b |(x^\nu - r_{\nu-1}(x))/\sqrt{Q(x)}|^q\ ((b-x)(x-a))^{-1/2}dx.$$

The statement of the theorem is an immediate consequence of Bernstein's result for $Q(x) = x^{2n}$, $\nu = 2n = 2m$ and the fact that $\tau_{2n}$ is the general Chebyshev polynomial associated with the Chebyshev space $S_n$ in $(a,b)$. The representation (4.1) follows from (2.5). $\Box$

We have already proved a Jackson type theorem for approximation by $S_n$-polynomials. An important question in Approximation Theory is to establish inverse type theorems. Basic tools for proving such theorems for approximation by algebraic polynomials are the inequalities of Markov [11] and Bernstein [1] (see also [12, 14]). It is known that Markov's and Bernstein's inequalities are sharp in the sence that there exist extremal polynomials for which equalities are attained. In fact, in both cases the extremal polynomial is the Chebyshev one. Chebyshev polynomials are solutions of some other interesting extremal problems concerning algebraic polynomials [14, Chapter 2B], [12, Chapter 6]. It is natural to expect that the Chebyshev $S_n$-polynomial $\tau_n$ plays a similar role in the corresponding problems concerning $S_n$-polynomials. We state our expectations in the following conjecture:

CONJECTURE 1 (MARKOV'S INEQUALITY) *For every $s \in S_n$*

$$\|s'\|_\infty \leq \frac{\sqrt{b}+\sqrt{a}}{2a(\sqrt{b}-\sqrt{a})}n^2\|s\|_\infty. \tag{4.2}$$

*Equality is attained if, and only if, $s = \tau_n$ and for $x = a$.*

CONJECTURE 2 (BERNSTEIN'S INEQUALITY) *For every $s \in S_n$ and any $x \in (a,b)$*

$$|s'(x)|_\infty \le \frac{x+\sqrt{ab}}{2x\sqrt{(b-x)(x-a)}} n\|s\|_\infty, \tag{4.3}$$

*where equality is attained only for $s = \tau_n$ at the zeros of $\tau_n$.*

CONJECTURE 3 (SZEGÖ'S INEQUALITY) *For every $s \in S_n$, for which $\|s\|_\infty \le 1$*

$$\frac{4(b-x)(x-a)x^2}{(x+\sqrt{ab})^2}(y')^2 + n^2y^2 \le n^2. \tag{4.4}$$

*Equality is attained if, and only if, $s = \tau_n$ when* (4.4) becomes the identity (2.5).

CONJECTURE 4 (BOJANOV-KRISTIANSEN'S INEQUALITY) *For any $p, 1 \le p \le \infty$ and every $s \in S_n$*

$$\|s'\|_p \le \|\tau_n'\|_p \|s\|_\infty, \tag{4.5}$$

*where $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are the uniform and $L_p$-norms in $(a,b)$, respectively. Equality is attained if, and only if, $s = \tau_n$.*

CONJECTURE 5 (THE LONGEST POLYNOMIAL) *Among all the $S_n$-polynomials $s$, for which $|s(x)| \le 1$ for $x \in [a,b]$, $\tau_n$ has the maximum length of arc.*

Inequalities (4.2) and (4.3) are analogs of the celebrated Markov's [11] and Bernstein's [1] inequalities, respectively. The polynomial version of inequality (4.4) was proved first by Szegö [16] and rediscovered later by van der Corput and Schaake [8]. Inequality (4.5) for algebraic polynomials was discovered independently by Bojanov [4] and Kristiansen [10]. Note that for $p = \infty$ (4.5) reduces to (4.2) and for $p = 1$ (4.5) can be verified just like in the polynomial case (see [4] and [14, p.149]). Bojanov [5] and Kristiansen [10] proved also that the Chebyshev polynomial $T_n$ is "the longest" one among all algebraic polynomials whose modulus is bounded by one in $[-1,1]$, thus verifying a long-standing conjecture of Erdös. For these and many other interesting extremal problems for algebraic polynomials we refer to [12] and [14].

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