Characterization of Generalized Bessel Polynomials in Terms of Polynomial Inequalities *

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ABSTRACT. *Generalized Bessel Polynomials (GBP) are characterized as the extremal polynomials in certain inequalities in L^2 norm of Markov type.*

1 Introduction and statement of results

Generalized Bessel polynomials $y_n(x;\alpha,\beta)$ are defined by

$$y_n(x;\alpha,\beta) := \sum_{k=0}^{n} (-n)_k (n+\alpha-1)_k \frac{(-x/\beta)^k}{k!} = {}_2F_0(-n, n+\alpha-1; -; -x/\beta), \tag{1}$$

where $(a)_k$ denotes the Pochhammer symbol, $(a)_0 := 1,\ (a)_k := a(a+1)\cdots(a+k-1),\ k \geq 1$. For $\alpha = \beta = 2$ they reduce to the simple Bessel polynomials $y_n(x) := y_n(x;2,2)$. A recent review of Srivastava [6] contains comprehensive historical information and new results about GBP. Here we mention only the following orthogonality relation given in [6].

Observe that for $\alpha,\beta \in \mathbb{R},\ \beta > 0,\ n \in \mathbb{N}_0$ and $2n < 1-\alpha$

$$\|p\|_{\alpha,\mathbb{P}_n} = \left(\int_0^\infty x^{\alpha-2} e^{-\beta/x} p^2(x)dx \right)^{1/2} \tag{2}$$

is a well-defined norm in the space $\mathbb{P}_n$ of real algebraic polynomials of degree not exceeding n. Then under the same restrictions on α,β and n, $\|p'\|_{\alpha+2,\mathbb{P}_{n-1}}$ is well-defined because the inequalities $2n < 1-\alpha$ and $2(n-1) < 1-(\alpha+2)$ are equivalent.

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First we prove an inequality of Markov type for the norm (2) for which GBP are the extremal polynomials:

Theorem 1 *Let $\alpha, \beta \in I\!R,\ \beta > 0,\ n \in I\!N_0$ and $2n < 1 - \alpha$. For every $p \in I\!P_n$ and any positive integer $k,\ k \leq n$*

$$\|p^{(k)}\|_{\alpha+2k, I\!P_{n-k}} \leq \left(\frac{n!}{(n-k)!} \prod_{j=1}^{k} (4 - \alpha - n - 3j) \right)^{1/2} \|p\|_{\alpha, I\!P_n}. \tag{3}$$

Moreover, for each $k,\ k = 1, \cdots, n$, equality is attained if and only if p is a constant multiple of $y_n(x; \alpha, \beta)$.

It is clear that the best constant $M_n(k, \alpha) := (n!/(n-k)!)^{1/2} \prod_{j=1}^{k} (4 - \alpha - n - 3j)^{1/2}$ in (3) is positive because $2n < 1 - \alpha$. Note that M_n depends on α but not on β. Inequalities similar to (3) which characterize the classical Jacobi and general Laguerre polynomials are obtained by the second author in [2] and independently by Guessab and Milovanović [3] and Min [5]. Chapter 6 of Milovanović, Mitrinović and Rassias's book [4] provides complete information about various inequalities for polynomials.

Next we shall prove an inequality of Markov type for a special type of "quasi-polynomials". Let $\alpha, \beta \in I\!R,\ \beta > 0,\ n \in I\!N_0$ and $2n < -1 - \alpha$. The set of "quasi-polynomials" $\mathcal{Q}_n(\alpha)$ is defined by

$$\mathcal{Q}_n(\alpha) := \left\{ q(x) = x^{\alpha} e^{-\beta/x} p(x) \ :\ p \in I\!P_n \right\}.$$

Then $\mathcal{Q}_n(\alpha)$ can be equipped with the norm

$$\|q\|_{\alpha, \mathcal{Q}_n(\alpha)} = \left(\int_0^{\infty} x^{-\alpha} e^{\beta/x} q^2(x) dx \right)^{1/2}. \tag{4}$$

Since for any $q \in \mathcal{Q}_n(\alpha)$ its derivative begongs to $\mathcal{Q}_{n+1}(\alpha - 2)$ and the restriction $2n < -1 - \alpha$ is equivalent to $2(n+1) < -1 - (\alpha - 2)$ then $\|q'\|_{\alpha, \mathcal{Q}_{n+1}(\alpha-2)}$ is well-defined.

Theorem 2 *Let $\alpha, \beta \in I\!R,\ \beta > 0,\ n \in I\!N_0$ and $2n < -1 - \alpha$. Then the inequality*

$$\|q'\|_{\alpha-2, \mathcal{Q}_{n+1}(\alpha-2)} \leq ((n+1)(-\alpha - n))^{1/2} \|q\|_{\alpha, \mathcal{Q}_n(\alpha)} \tag{5}$$

holds for every $q \in \mathcal{Q}_n(\alpha)$. Moreover, equality is attained if and only if q is a constant multiple of $x^{\alpha} e^{-\beta/x} y'_{n+1}(x; \alpha, \beta)$.

Note that the expression $(n+1)(-\alpha - n)$ which appears on the right-hand side of (5) is positive because $2n < -1 - \alpha$.

2 Proofs

It is well known that the generalized Bessel polynomial of degree n satisfies the second order differential equation

$$x^2 y'' + (\alpha x + \beta) y' - n(n + \alpha - 1) y = 0. \tag{6}$$

Obviously this can be rewritten in the self-adjoint form:

$$(x^{\alpha}e^{-\beta/x}y')' = n(n+\alpha-1)x^{\alpha-2}e^{-\beta/x}y. \tag{7}$$

It follows immediately from (1) that

$$y_n'(x;\alpha,\beta) = \frac{n(n+\alpha-1)}{\beta}y_{n-1}(x;\alpha+2,\beta). \tag{8}$$

Proof of Theorem 1. First we prove (3) for $k=1$. Let p be an arbitrary polynomial of degree n. Under the restrictions on α,β and n the polynomials $y_0(x;\alpha,\beta),\cdots,y_n(x;\alpha,\beta)$ are orthogonal polynomials. Hence they form a basis in $I\!P_n$ and p can be uniquely represented in the form $p(x)=\sum_{j=0}^{n}a_j\tilde{y}_j(x;\alpha,\beta)$, where

$$\tilde{y}_j(x;\alpha,\beta) := \left(\frac{1-\alpha-2j}{\beta^{\alpha-1}j!\Gamma(2-\alpha-j)}\right)^{1/2} y_j(x;\alpha,\beta), \quad j=0,\cdots,n$$

are the orthonormalized GBP. Therefore

$$||p||^2_{\alpha,I\!P_n} = \sum_{j=0}^{n} a_j^2.$$

Since $p'(x)=\sum_{j=1}^{n}a_j\tilde{y}_j'(x;\alpha,\beta)$ then (8) yields

$$p'(x) = \sum_{j=1}^{n} a_j \left(\frac{1-\alpha-2j}{\beta^{\alpha-1}j!\Gamma(2-\alpha-j)}\right)^{1/2} \frac{j(j+\alpha-1)}{\beta}y_{j-1}(x;\alpha+2,\beta).$$

Note that the polynomials $y_0(x;\alpha+2,\beta),\cdots,y_{n-1}(x;\alpha+2,\beta)$ are orthogonal on $(0,\infty)$ with respect to the weight function $x^{\alpha}e^{-\beta/x}$. The requirement $\alpha<1-2n$ is equivalent to $\alpha+2<1-2(n-1)$. Therefore we can apply (**??**) for $\alpha=\alpha+2$ and $m,j\le n-1$. Thus

$$\begin{aligned} ||p'||^2_{\alpha+2,I\!P_{n-1}} &= \sum_{j=1}^{n} \frac{1-\alpha-2j}{\beta^{\alpha-1}j!\Gamma(2-\alpha-j)}\frac{j^2(j+\alpha-1)^2}{\beta^2}\beta^{\alpha+1}\frac{(j-1)!}{1-\alpha-2j}\Gamma(1-\alpha-j)a_j^2 \\ &= \sum_{j=1}^{n} j(1-\alpha-j)a_j^2. \end{aligned}$$

The theorem will be proved if we show that the solution of the extremal problem

$$\max\{\sum_{j=0}^{n} j(1-\alpha-j)a_j^2/\sum_{j=0}^{n}a_j^2 \ : \ a_0,\cdots,a_n\in I\!R,\ \sum_{j=0}^{n}a_j^2\neq 0\} \tag{9}$$

is $(M_n(1,\alpha))^2$ and it is attained for $a_0=\cdots=a_{n-1}=0$. It is well-known and easy to see that (9) is equivalent to the problem of determination

$$\max_{1\le j\le n} j(1-\alpha-j).$$

Since $f(j) := j(1-\alpha-j)$ is a concave binomial with zeros at 0 and $1-\alpha$ then f is an increasing function of j for $j \in (0,(1-\alpha)/2)$. On the other hand the requirement $\alpha < 1-2n$ is equivalent to $n < (1-\alpha)/2$. Hence $f(j) < f(n)$ for $j = 0,\cdots,n-1$. Therefore the solution of the extremal problem (9) is $n(1-\alpha-n) = (M_n(1,\alpha))^2$ and the maximal value is attained only for $a_0 = \cdots = a_{n-1} = 0$ and an arbitrary non-zero a_n. This completes the proof for $k=1$. The inequalities (3) for $k = 2,\cdots,n$ follow by repeated application of that for $k=1$.

Proof of Theorem 2. Let $q \in \mathcal{Q}_n(\alpha)$ be arbitrary. Since the polynomials $y_1'(x;\alpha,\beta),\cdots,$ $y_{n+1}'(x;\alpha,\beta)$ form a basis in $\mathcal{Q}_n(\alpha)$ then q can be uniquely represented in the form $q(x) = x^\alpha e^{-\beta/x}\sum_{j=1}^{n+1} a_j y_j'(x;\alpha,\beta)$. Having in mind that under the restrictions α,β and n the polynomials $y_j'(x;\alpha,\beta)$, $j = 1,\cdots,n+1$ are orthogonal on $(0,\infty)$ with respect to the weight function $x^\alpha e^{-\beta/x}$ and using (8) and (**??**) for $\alpha = \alpha+2$ we get

$$\begin{aligned}
\|q\|^2_{\alpha,\mathcal{Q}_n} &= \int_0^\infty x^\alpha e^{-\beta/x}\left(\sum_{j=1}^{n+1} a_j y_j'(x;\alpha,\beta)\right)^2 dx \\
&= \sum_{j=1}^{n+1} a_j^2 \|y_j'(x;\alpha,\beta)\|^2_{\alpha+2,I\!P_{n-1}} \qquad (10)\\
&= \sum_{j=1}^{n+1} a_j^2 \beta^{\alpha-1} j^2 (j+\alpha-1)^2 \frac{(j-1)!}{1-\alpha-2j}\Gamma(1-\alpha-j).
\end{aligned}$$

On the other hand, (7) yields

$$q'(x) = x^{\alpha-2}e^{-\beta/x}\sum_{j=1}^{n+1} a_j j(j+\alpha-1) y_j(x;\alpha,\beta).$$

Then

$$\begin{aligned}
\|q'\|^2_{\alpha-2,\mathcal{Q}_{n+1}} &= \int_0^\infty x^{\alpha-2} e^{-\beta/x}\left(\sum_{j=1}^{n+1} a_j j(j+\alpha-1) y_j(x;\alpha,\beta)\right)^2 dx \\
&= \sum_{j=1}^{n+1} a_j^2 \beta^{\alpha-1} j^2 (j+\alpha-1)^2 \frac{j!}{1-\alpha-2j}\Gamma(2-\alpha-j). \qquad (11)
\end{aligned}$$

Introducing the notation

$$b_j^2 := a_j^2 \beta^{\alpha-1} j^2 (j+\alpha-1)^2 \frac{(j-1)!}{1-\alpha-2j}\Gamma(1-\alpha-j)$$

we can rewrite (10) and (11) in the form

$$\|q\|^2_{\alpha,\mathcal{Q}_n} = \sum_{j=1}^{n+1} b_j^2$$

and

$$\|q'\|^2_{\alpha-2,\mathcal{Q}_{n+1}} = \sum_{j=1}^{n+1} j(1-\alpha-j)b_j^2,$$

respectively. Note that the requirements of the theorem are equivalent to $n+1 < (1-\alpha)/2$. Then the same reasoning as in the proof of Theorem 1 completes the proof.

References

[1] P. P. Agarwal and G. V. Milovanivić, One characterization of the classical orthogonal polynomials, in "Progress in Approximation Theory," (P. Nevai and A. Pinkus, Eds.), Academic Press, New York, 1991, 1-4.

[2] D. K. Dimitrov, Bernstein inequality in L_2 norm, *Math. Balkanika (N. S.)* **7**(1993), 131-136.

[3] A. Guessab and G. V. Milovanivić, Weighted L^2-analogues of Bernstein's inequality and classical orthogonal polynomials, *J.Math.Anal.Appl.*, **182**(1994), 244-249.

[4] G. V. Milovanivić, D. S. Mitrinović and Th. M. Rassias, "Topics in Polynomials: Extremal Problems, Inequalities, Zeros," World Scientific, Singapore, 1994.

[5] G. H. Min, Bernstein-Markov inequalities in L^2 spaces and their optimal constants.II, *J. Math. Res. Exposition* **14**(1994), 135-138.

[6] H. M. Srivastava, Orthogonality relations and generating functions for the generalized Bessel polynomials, *Appl. Math. Comp.* **61**(1994), 99-134.

[7] A. K. Varma, A new characterization of Hermite polynomials, *Acta Math. Hung.* **49**(1987), 169-172.