Electrostatic Interpretation and Bounds for the Extreme Zeros of Classical Orthogonal Polynomials *

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Abstract. *Let* $P_n^{(\alpha,\beta)}$, $n = 0, 1, \ldots,$ $\alpha, \beta > -1$ *be the Jacobi polynomials, orthogonal on* $(-1, 1)$ *with respect to the weight function* $(1-x)^{\alpha}(1+x)^{\beta}$. *Denote by* $x_{n,k}(\alpha, \beta), k = 1, \ldots, n$ *the zeros of* $P_n^{(\alpha,\beta)}$ *enumerated in decreasing order. Lower bound for* $x_{n,n}(\alpha, \beta)$ *and upper bound for* $x_{n,1}(\alpha, \beta)$ *are obtained as a consequence of the electrostatic interpretation of the zeros of* $P_n^{(\alpha,\beta)}$. *The results are applied then to obtain bounds for the extreme zeros of Laguerre and Hermite polynomials.*

1 Introduction

One of the important reasons for the interest in the zeros of Jacobi polynomials is the fact that $x_{n,k}(\alpha, \beta), k = 1, \ldots, n$ admit a very interesting electrostatic interpretation. They are the points of equilibrium of n unit charges in $(-1, 1)$ in the field generated by charges $(\alpha + 1)/2$ at 1 and $(\beta + 1)/2$ at -1, where the charges repell each other according to the law of logarithmic potential. It is worth mentioning that this electrostatic field is not the field of point charges but of long straight wires perpendicular to the real axis [9]. This beautiful interpretation is due to Stieltjes [10, 11, 12]. Szegö [13, Section 6.7] proved that the equilibrium is a minimum of the energy of the described field. We refer to the review papers [14, 15] for the historical background and other results concerning electrostatics of zeros of orthogonal polynomials.

Thus, any result about the behaviour of $x_{n,k}(\alpha, \beta)$ provides information about the above electrostatic field. In this paper, the opposite approach is suggested. We employ the above results of Stielties and Szegö to obtain bounds for the extreme zeros of Jacobi polynomials.

For every positive integer n and for any real $\alpha > -1$ and $\beta > -1$, the function $B(n, \alpha, \beta)$ is defined by

$$\begin{aligned} B(n,\alpha,\beta) \;\; = \;\; & (\beta^2 - \alpha^2)^2 + \\ & (4n(n+\alpha+\beta+1) + (\alpha+\beta)(\alpha+\beta+2)) \\ & \left(4n(n+\alpha+\beta+1) + 2(\alpha+\beta+2) - (\beta-\alpha)^2\right). \end{aligned}$$

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THEOREM 1 *For every positive integer n and any real $\alpha > -1$ and $\beta > -1$ the inequalities*

$$x_{n,n}(\alpha,\beta) > \frac{\beta^2 - \alpha^2 - \sqrt{B(n,\alpha,\beta)}}{4n(n+\alpha+\beta+1) + (\alpha+\beta)(\alpha+\beta+2)} \tag{1.1}$$

and

$$x_{n,1}(\alpha,\beta) < \frac{\beta^2 - \alpha^2 + \sqrt{B(n,\alpha,\beta)}}{4n(n+\alpha+\beta+1) + (\alpha+\beta)(\alpha+\beta+2)} \tag{1.2}$$

hold.

As consequences we obtain bounds for the extreme zero of Laguerre and Hermite polynomials.

COROLLARY 1 *Let $h_{n,1}$ be the largest zero of the Hermite polynomial $H_n(x)$. Then*

$$h_{n,1} \leq \sqrt{2n-1}. \tag{1.3}$$

COROLLARY 2 *Let $x_{n,n}(\alpha)$ and $x_{n,1}(\alpha)$ be the smallest and largest zero Laguerre polynomial $L_n^{(\alpha)}(x)$, respectively. Then for any $\alpha > -1$ the inequalities*

$$x_{n,n}(\alpha) \geq 2n + \alpha + 1 - \sqrt{(2n+\alpha+1)^2 + 1 - \alpha^2} \tag{1.4}$$

and

$$x_{n,1}(\alpha) \leq 2n + \alpha + 1 + \sqrt{(2n+\alpha+1)^2 + 1 - \alpha^2} \tag{1.5}$$

hold.

Although the bounds obtained in Theorem 1 and Corollaries 1 and 2 are not the sharpest one, they are asymptoticaly sharp in the sence that as n diverges, they behave like the best known bounds. Sharper bounds for the extreme zeros of Jacobi polynomials are given by Elbert, Laforgia and Rodonó [3] and by Ismail and Li [7]. The best lower bound for the smallest zero of Laguerre polynomial is that obtained by Ismail and Li [7] and the best upper bound for the largest zero of Laguerre polynomial is given by Szegö [13, Theorem 6.32]. Inequality (1.3) is slightly sharper than the inequality $h_{n,1} < \sqrt{2n}$, which was obtained recently by Ifantis and Siafarikas [6].

2 PROOF OF THEOREM 1

Consider the electostatic field, described above. Charges $(\alpha+1)/2$ and $(\beta+1)/2$ are distributed along long wires wich are perpendicular to the real axes and intersect it at 1 and -1, respectively. The remaining n unit charges are distributed along long wires which are also perpendicular to the real axes and pass through the points $x_1, \ldots, x_n$, where $-1 < x_1 < \ldots < x_n < 1$. Then the energy of the field is given by

$$L(x_1,\ldots,x_n) = \frac{\alpha+1}{2}\sum_{i=1}^{n}\log\frac{1}{|1-x_i|} + \frac{\beta+1}{2}\sum_{i=1}^{n}\log\frac{1}{|1+x_i|} + \sum_{1\leq i<j\leq n}\log\frac{1}{|x_j - x_i|}.$$

It is clear that the energy is minimal if and only if the function

$$T(x_1,\ldots,x_n) := \prod_{i=1}^{n}(1-x_i)^{(\alpha+1)/2}(1+x_i)^{(\beta+1)/2}\prod_{1\le i<j\le n}|x_j-x_i|. \tag{2.1}$$

is maximal. Stiltjes and Szegö's results are summarized in the following theorem:
THEOREM A ([13, Theorem 6.7.1]) *Let $\alpha>-1,\ \beta>-1$, and let $\{x_i\},\ -1\le x_i\le 1$, be a system of values for which the expression* (2.1) *becomes a maximum. Then the $\{x_i\}$ are the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$.*

The proof is straightforward and uses the fact that $y(x)=P_n^{(\alpha,\beta)}(x)$ is the unique (up to a constant factor) nonzero polynomial solution of the differential equation

$$(1-x^2)y''+(\beta-\alpha-(\alpha+\beta+2)x)y'+n(n+\alpha+\beta+1)y=0. \tag{2.2}$$

More precisely, one needs the following theorem:

THEOREM B ([13, Theorem 4.2.2]) *Let $\alpha>-1$ and $\beta>-1$. The differential equation*

$$(1-x^2)y''+(\beta-\alpha-(\alpha+\beta+2)x)\,y'+\gamma y=0,$$

where γ is a parameter, has a polynomial solution not identically zero if and only if γ has the form $n(n+\alpha+\beta+1),\ n=0,1,\ldots.$ This solution is $\mathrm{const}P_n^{(\alpha,\beta)}(x)$, and no solution which is linearly independent of $P_n^{(\alpha,\beta)}(x)$ can be a polynomial.

It is well-known (see [13, p. 117]) and easy to see that for any nonnegative integer l, $y(x)=P_n^{(\alpha,\beta)}(x)$ is a solution of the differential equation

$$(1-x^2)y^{(l+2)}+(\beta-\alpha-(\alpha+\beta+2l+2)x)\,y^{(l+1)}+(n(n+\alpha+\beta+1)-l(l+\alpha+\beta+1))\,y^{(l)}=0. \tag{2.3}$$

Proof of Theorem 1. Consider the function $T^2(x_1,\ldots,x_n)$ on the simplex $\Xi:=\{-1<x_1<\ldots<x_n<1\}$. Observe that

$$\begin{aligned}T^2(x_1,\ldots,x_n) &= \tau(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)(1-x_k)^{\alpha+1}(1+x_k)^{\beta+1}\prod_{i\ne k}(x_k-x_i)^2\\ &= \tau(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)(1-x_k)^{\alpha+1}(1+x_k)^{\beta+1}\omega_k^2(x_k).\end{aligned}$$

Here the function $\tau(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)=\tau(\bar{x}\setminus x_k)$ does not depend on x_k and $\omega_k(x):=\omega(x)/(x-x_k)$, where $\omega(x):=\prod_{i=1}^{n}(x-x_i)$.

Differentiating the latter identity and using the equalities $\omega^{(l+1)}(x_k)=(l+1)\omega_k^{(l)}(x_k),\ l=0,1,2,$ we obtain consecutively

$$\begin{aligned}\frac{\partial}{\partial x_k}T^2(x_1,\ldots,x_n) &= \tau(\bar{x}\setminus x_k)\frac{\partial}{\partial x_k}\Big((1-x_k)^{\alpha+1}(1+x_k)^{\beta+1}\omega_k^2(x_k)\Big)\\ &= \tau(\bar{x}\setminus x_k)\ (1-x_k)^{\alpha}(1+x_k)^{\beta}\omega_k(x_k)\\ &\quad \{2(1-x_k^2)\omega_k'(x_k)+(\beta-\alpha-(\alpha+\beta+2)x_k)\,\omega_k(x_k)\}\\ &= \tau(\bar{x}\setminus x_k)(1-x_k)^{\alpha}(1+x_k)^{\beta}\omega'(x_k)\\ &\quad \{(1-x_k^2)\omega''(x_k)+(\beta-\alpha-(\alpha+\beta+2)x_k)\,\omega'(x_k)\}\,.\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
\tau^{-1}(\bar{x} \setminus x_k)\frac{\partial^2}{\partial x_k^2}T^2(x_1,\ldots,x_n) &= \frac{\partial}{\partial x_k}\Big((1-x_k)^\alpha(1+x_k)^\beta\omega_k(x_k)\Big) \\
&\quad \{2(1-x_k^2)\omega_k'(x_k)+(\beta-\alpha-(\alpha+\beta+2)x_k)\,\omega_k(x_k)\} \\
&\quad + (1-x_k)^\alpha(1+x_k)^\beta\omega_k(x_k) \\
&\quad \{2(1-x_k^2)\omega_k''(x_k)+(\beta-\alpha-(\alpha+\beta+6)x_k)\omega_k'(x_k) \\
&\quad -(\alpha+\beta+2)\omega_k(x_k)\} \\
&= \frac{\partial}{\partial x_k}\Big((1-x_k)^\alpha(1+x_k)^\beta\omega_k(x_k)\Big) \\
&\quad \{(1-x_k^2)\omega''(x_k)+(\beta-\alpha-(\alpha+\beta+2)x_k)\,\omega'(x_k)\} \\
&\quad + (1-x_k)^\alpha(1+x_k)^\beta\omega'(x_k) \\
&\quad \Big\{(1-x_k^2)\tfrac{2\omega'''(x_k)}{3}+(\beta-\alpha-(\alpha+\beta+6)x_k)\tfrac{\omega''(x_k)}{2} \\
&\quad -(\alpha+\beta+2)\omega'(x_k)\Big\}\,.
\end{aligned} \tag{2.5}$$

The function T^2 is positive in Ξ and vanishes on the boundary of Ξ. Therefore the maximal value is attained at an intrinsic point of Ξ. This means that the maximum is local. Now (2.4) and Theorem B complete the proof of Theorem A.

Another implication of the fact that the maximum of T^2 is local and it is unique, is that $\partial^2T^2/\partial x_k^2$ are negative for $k=1,\ldots,n$ at the only extremal point $(x_{n,n}(\alpha,\beta),\ldots,x_{n,1}(\alpha,\beta))$. Since $\tau^{-1}(\bar{x}\setminus x_k) > 0$ and $(1-x_k^2)\omega''(x_k)+(\beta-\alpha-(\alpha+\beta+2)x_k)\,\omega'(x_k)=0$ for $x_k=x_{n,k}(\alpha,\beta),\quad k=1,\ldots,n$, then

$$\begin{aligned}
F(x_k) &:= \omega'(x_k)\left\{\frac{2}{3}(1-x_k^2)\omega'''(x_k)+\frac{1}{2}(\beta-\alpha-(\alpha+\beta+6)x_k)\omega''(x_k)-(\alpha+\beta+2)\omega'(x_k)\right\} \\
&< 0, \quad \textit{for} \;\; x_k=x_{n,k}(\alpha,\beta), \quad k=1,\ldots,n.
\end{aligned}$$

We use (2.3) for $l=1$ and $l=0$ consecutively in order to simplify $F(x_k)$:

$$\begin{aligned}
6F(x_k) &= \omega'(x_k)\quad \{[(\alpha+\beta-2)x_k+\alpha-\beta]\omega''(x_k) \\
&\qquad -[4n(n+\alpha+\beta+1)+2(\alpha+\beta+2)]\omega'(x_k)\} \\
&= \frac{(\omega'(x_k))^2}{1-x_k^2}\quad \{[(\alpha+\beta-2)x_k+\alpha-\beta]\;[(\alpha+\beta+2)x_k+\alpha-\beta] \\
&\qquad -[4n(n+\alpha+\beta+1)+2(\alpha+\beta+2)]\;(1-x_k^2)\}\,.
\end{aligned}$$

Therefore the inequalities

$$\begin{aligned}
&[(\alpha+\beta-2)x_k+\alpha-\beta]\;[(\alpha+\beta+2)x_k+\alpha-\beta] \\
&\quad -[4n(n+\alpha+\beta+1)+2(\alpha+\beta+2)]\;(1-x_k^2)\quad <0
\end{aligned}$$

hold for $x_k=x_{n,k}(\alpha,\beta),\quad k=1,\ldots,n$. This latter statement is equivalent to the statement of the theorem. □

3 Bounds for the extreme zeros of Gegenbauer, Laguerre and Hermite polynomials

Let $\zeta_{n,1}(\lambda)$ be the largest zero of the Gegenbauer (ultraspherical) polynomial C_n^λ. There are three recent results about upper bounds for $\zeta_{n,1}(\lambda)$. We refer to [8] for a more detailed review. Elbert and

Laforgia [1, Theorem 1 (ii)] proved that

$$\zeta_{n,1}(\lambda) < \sqrt{\frac{n^2+2\lambda n}{n^2+2\lambda n+\lambda^2+\lambda}} \quad for \quad \lambda \geq 0,$$

and Ifantis and Siafarikas [5] proved the inequality

$$\zeta_{n,1}(\lambda) < \sqrt{\frac{n^2+2\lambda n-n}{n^2+2\lambda n+\lambda^2-n-\lambda}} \quad for \quad \lambda > 1/2.$$

Förster and Petras [4] obtained the upper bound

$$\zeta_{n,1}(\lambda) < \sqrt{\frac{n^2+2\lambda n-\lambda-1/2}{n^2+2\lambda n+\lambda^2-1/4}} \quad for \quad \lambda > -1/2. \tag{3.1}$$

Substituting $\alpha = \beta = \lambda - 1/2$ in (1.2), we obtain

$$\zeta_{n,1}(\lambda) < \sqrt{\frac{n^2+2\lambda n+\lambda+1/2}{n^2+2\lambda n+\lambda^2-1/4}} \quad for \quad \lambda > -1/2. \tag{3.2}$$

Although our estimate (3.2) is obviously weaker than (3.1), its advantage is that it is an immediate consequence of an estimate for $x_{n,1}(\alpha,\beta)$, while all the previous results hold specifically for the largest zeros of ultraspherical polynomials. We employ the inequality (3.1) in order to give a very short proof of Corollary 1.

Proof of Corollary 1. It was pointed out by Elbert and Laforgia [2] that for any positive integer n

$$\sqrt{\lambda}\zeta_{n,1}(\lambda) \longrightarrow h_{n,1} \quad as \quad \lambda \to \infty.$$

Then multiplying (3.1) by $\sqrt{\lambda}$ and letting λ to diverge, we obtain the desired result. □

Proof of Corollary 2. Let $A(n,\alpha,\beta)$ be defined by

$$A(n,\alpha,\beta) = 4n(n+\alpha+\beta+1) + (\alpha+\beta)(\alpha+\beta+2) + \alpha^2 - \beta^2.$$

Then inequalities (1.1) and (1.2) yield

$$1 - x_{n,n}(\alpha,\beta) < \frac{A(n,\alpha,\beta)+\sqrt{B(n,\alpha,\beta)}}{4n(n+\alpha+\beta+1)+(\alpha+\beta)(\alpha+\beta+2)} \tag{3.3}$$

and

$$1 - x_{n,1}(\alpha,\beta) > \frac{A(n,\alpha,\beta)-\sqrt{B(n,\alpha,\beta)}}{4n(n+\alpha+\beta+1)+(\alpha+\beta)(\alpha+\beta+2)}, \tag{3.4}$$

respectively. On the other hand, by (6.71.11) in [13], we have

$$\lim_{\beta\to\infty} \frac{\beta}{2}(1-x_{n,n}(\alpha,\beta)) = x_{n,1}(\alpha) \tag{3.5}$$

and

$$\lim_{\beta\to\infty} \frac{\beta}{2}(1-x_{n,1}(\alpha,\beta)) = x_{n,n}(\alpha). \tag{3.6}$$

Since

$$A(n,\alpha,\beta)=2\Big((2n+\alpha+1)\beta+2n(n+\alpha+1)+\alpha^2+\alpha\Big)$$

and $B(n,\alpha,\beta)$ is a quadratic polynomial in β with leading coefficient $4\left((2n+\alpha+1)^2+1-\alpha^2\right)$, then the relations (3.3) and (3.5) imply (1.5). Similarly, (3.4) and (3.6) yield (1.4). $\Box$

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