ASYMPTOTICS OF ZEROS OF POLYNOMIALS ARISING FROM RATIONAL INTEGRALS

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Abstract. We prove that the zeros of the polynomials $P_m(a)$ of degree m, defined by Borosh and Moll via

$$P_m(a) = \frac{2^{m+3/2}}{\pi}(a+1)^{m+1/2}\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}},$$

approach the lemniscate $\{\zeta \in \mathbb{C} \ : \ |\zeta^2 - 1| = 1, \ \Re\zeta < 0\}$, as m diverges.

1. Introduction

Recently Moll, Borosh and their coauthors [1, 2, 3, 6] investigated thoroughly the rational integral

$$N(a,m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, \tag{1}$$

its dependence both on the parameters m and a, and established various interesting properties of $N(a,m)$. We refer to a recent personal account of Moll [6] about the motivation for the study of the integrals (1) and about some unexpected interplays. Among the other facts, they proved that, for every fixed $m \in \mathbb{N}$,

$$P_m(a) = \frac{2^{m+3/2}}{\pi}(a+1)^{m+1/2}N(a,m)$$

is a polynomial in a of degree m. It was observed in [1] that the zeros of $P_m(a)$ possess a pretty regular asymptotic behaviour and nice pictures in support of this observation were furnished in [1, 6]. It was pointed out by Borosh and Moll in [1], that their numerical experiments suggest that the zeros go to a lemniscate but they were not able to predict its equation. We prove that the limit curve for the zeros of $P_m(a)$ is the left half of the lemniscate of Bernoulli

$$L = \{\zeta \in \mathbb{C} \ : \ |\zeta^2 - 1| = 1, \ \Re\zeta < 0\}.$$

The polar equation of L is

$$\rho^2 = 2\cos 2\theta, \quad \theta \in (3\pi/4, 5\pi/4).$$

We also try to explain the lack of zeros around the part of L which is close to the origin, a fact which can be observed in the above mentioned pictures and in the Figure 1 below which shows the zeros of $P_{70}(a)$.

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Theorem 1. *If Z_m is the set of the zeros of $P_m(a)$, then*

$$\max_{a\in Z_m} \min\{|a-\zeta| \quad : \quad \zeta\in\mathbb{C},\ |\zeta^2-1|=1,\ \Re\zeta<0\} \longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

Moreover, if $m\in\mathbb{N}$ is fixed, then

$$|a+1| \le 1 - \frac{2\left(\sqrt{m(m+1)(4m+1)}-m\right)}{(2m+1)^2} \quad \text{for every } a\in Z_m.$$

The zeros of $P_{70}(z)$, together with the lemniscate L and the circumference

$$C_m = \left\{\zeta\in\mathbb{C} \quad : \quad |\zeta+1| = 1 - \frac{2\left(\sqrt{m(m+1)(4m+1)}-m\right)}{(2m+1)^2}\right\},$$

for $m=70$, can be seen in Figure 1.

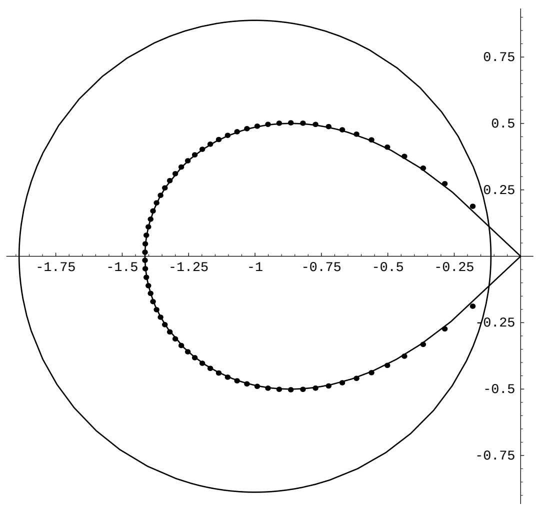

FIGURE 1. The zeros of $P_{70}(a)$, the lemniscate L and the circumference C_{70}.

2. Proof of the theorem

We begin with a simple technical lemma which shows that the inverse polynomial of a hypergeometric polynomial is also hypergeometric. Recall that the hypergeometric function is defined by

$$F(\alpha,\beta;\gamma;x) := {}_2F_1(\alpha,\beta;\gamma;x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!},$$

where $(\alpha)_k$ denotes the Pochhammer symbol, $(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1)$, for $k\in\mathbb{N}$, and $(\alpha)_0 := 1$.

Lemma 1. *If $n\in\mathbb{N}$, $\beta,\gamma\in\mathbb{C}$, with $\beta,\gamma\neq 0,-1,-2,\ldots,-n$, then*

$$F(-n,\beta;\gamma;z) = \frac{(\beta)_n}{(\gamma)_n}(-z)^n F(-n,1-\gamma-n;1-\beta-n;1/z). \tag{2}$$

Proof. Calculating the coefficient d_k of z^k on the right-hand side of (2), we obtain

$$\begin{aligned} d_k &= (-1)^n \frac{(\beta)_n}{(\gamma)_n} \frac{(-n)_{n-k}(1-\gamma-n)_{n-k}}{(1-\beta-n)_{n-k}} \frac{1}{(n-k)!} \\ &= \frac{(-1)^k}{k!} \frac{(\beta)_k}{(\gamma)_k} n(n-1)\cdots(n-k+1) \\ &= \frac{(\beta)_k}{(\gamma)_k}(-n)_k \frac{1}{k!} \end{aligned}$$

and the latter coincides with the coefficient of z^k on the left-hand side of (2). □

We shall need the following version of the classical Eneström-Kakeya theorem (see Excercise 2 on p. 137 in [5]):

Theorem A. *All the zeros of the polynomial $f(z) = c_0 + c_1 z + \cdots + c_n z^n$, having positive coefficients c_j, lie in the ring*

$$\min_{0 \le k \le n-1} (c_k/c_{k+1}) \le |z| \le \max_{0 \le k \le n-1} (c_k/c_{k+1}).$$

Proof of Theorem 1. It was proved in [2, 3] that $P_m(a)$ is a hypergeometric polynomial,

$$P_m(a) = \binom{2m}{m} F(-m, m+1; 1/2-m; (a+1)/2)).$$

On using this representation and applying identity (2) for $n = m$, $\beta = m+1$, $\gamma = 1/2 - m$ and $z = (a+1)/2$, we obtain

$$P_m(a) = (-1)^m \binom{2m}{m} \frac{(m+1)_m}{(1/2-m)_m} \left(\frac{a+1}{2}\right)^m F\left(-m, \frac{1}{2}; -2m; \frac{2}{a+1}\right). \tag{3}$$

On the other hand, Driver and Möller [4] proved that, for any real positive b, the zeros of the polynomials

$$w^n F(-n, b; -2n; 1/w) \tag{4}$$

approach the Cassini curve

$$|(2w-1)^2 - 1| = 1 \tag{5}$$

as n diverges. Their result and the representation (3) immediately imply that the zeros of $P_m(a)$ tend to the lemniscate $|\zeta^2 - 1| = 1$ as m goes to infinity. In order to prove that these zeros remain in the disc D_m surrounded by the circumference C_m, we apply Theorem A. The coefficients c_k in the expansion

$$F(-m, m+1; 1/2-m; (a+1)/2)) = \sum_{k=0}^{m} c_k (a+1)^k$$

are positive. Moreover

$$\Delta(k) := \frac{c_k}{c_{k+1}} = \frac{(k+1)(2m-2k-1)}{(m-k)(m+k+1)}.$$

Straightforward calculations show that,

$$\Delta'(\kappa) = \frac{(2m-1)\kappa^2 - 2(2m^2+1)\kappa + 2m^3 - m^2 - m - 1}{(m-\kappa)^2(m+\kappa+1)^2},$$

and $\Delta'(\kappa) = 0$ only for

$$\kappa_{1,2} = \frac{2m^2 + 1 \pm \sqrt{m(m+1)(4m+1)}}{2n-1},$$

where $0 < \kappa_1 < m-1 < \kappa_2$. It is easy to see that κ_1 is a point of local and also of global maximum of $\Delta(\kappa)$ when κ varies in $[0, m-1]$. Then, by Theorem A we conclude that the zeros of $P_m(a)$ lie in the disc $|a+1| \le \Delta(\kappa_1)$, where

$$\Delta(\kappa_1) = 1 - \frac{2\left(\sqrt{m(m+1)(4m+1)} - m\right)}{(2m+1)^2}.$$

This completes the proof of the theorem.

3. Remarks and open questions

While the proof of the convergence of the zeros of $P_m(a)$ to L is a matter of simple transformation of Driver and Möller's result, the lack of zeros close to the origin seems to be an interesting phenomenon. Except for the fact that Z_m is a subset of D_m, one may obtain other regions which do not contain zeros of P_m.

An application of a generalization of Descartes' rule of signs, due to Obrechkoff [7] (see Theorem 41.3 in [5]), implies

$$|\arg(a+1)| \ge \pi/m \ \text{ for every } a \in Z_m.$$

In other words, the zeros of $P_m(a)$ are outside the infinite sector with vertex -1 which contains the ray $(-1, \infty)$, with angle $2\pi/m$.

Another method which is applicable to our case is the so-called Parabola theorem of Saff and Varga [8]. It guarantees that the zeros of certain polynomials that satisfy a three-term recurrence relation belong to a parabola region. It can be shown that the polynomials

$$q_k(z) = \frac{(1/2-m)_k}{(-1/2-2m)_k} F(-k, m+1, 1/2-m, 1+z)$$

satisfy the recurrence relation

$$q_k(z) = \left(\frac{z}{b_k} + 1\right) q_{k-1}(z) - \frac{z}{c_k} q_{k-2}(z), \quad k = 1, \ldots, m,$$

where

$$b_k = \frac{2m-k+3/2}{m+k} \quad \text{and} \quad c_k = \frac{(2n-k+3/2)(2n-k+5/2)}{(k-1)(n-k+3/2}.$$

Then if $a \in Z_m$, with $a = \xi + i\eta$, the parabola theorem yields

$$\eta^2 > \frac{2(4m+1)}{2m-1}\left(\xi + \frac{3}{4m-2}\right).$$

Let us apply the left-hand side estimate in Theorem A to the Maclaurin expansion

$$P_m(a) = \sum_{k=0}^{m} d_k(m) a^k.$$

In [1] Borosh and Moll obtained the representation

$$d_k(m) = 2^{-2m} \sum_{j=k}^{m} 2^j \binom{2m-2j}{m-j}\binom{m+j}{m}\binom{j}{k}$$

for the coefficients of this expansion and proved that the first half of the sequence $\{d_k(m)\}_{k=0}^m$ is increasing and the second one is decreasing. Numerical experiments show that, for every fixed m, the minimum of $d_k(m)/d_{k+1}(m)$ is attained for $k = 0$ and the values of $d_0(m)/d_1(m)$ behave like $1/m$ as m diverges.

Summarizing, these application of the results of Obrechkoff, of Saff and Varga and the Eneström-Kakeya theorem guarantee the lack of zeros in very "tiny" regions around the origin. More precisely, the circumference C_m cuts a larger portion of the lemniscate L around the origin in comparison with the cuts with the above mentioned sector, parabola and circumference of radius $1/m$ around the origin. The author finds that a promising way of determining the precise behaviour of the closest to the origin zero of $P_m(a)$ is the relation between Z_m and the zeros of the polynomials $H_n(b, u)$, defined by (4.4) in [4]. There, on p. 86, Driver and Möller formulated a challenging conjecture about the location of the zeros of $H_n(b, u)$. If true, and once it is established, the statement of that conjecture will provide precise information about the location of the zeros of $P_m(a)$.

We finish with an observation motivated by another result of Driver and Möller. Proposition 5.2 in [4] states that the zeros of the polynomials (4) lie outside the Cassini curve (5) for all $n \in \mathbb{N}$, provided $b \geq 1$. In our case $b = 1/2$, so we can not conclude that the zeros of $P_m(a)$ lie outside L though numerical experiments show that they do.

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