Generating Starlike and Convex Univalent Functions

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Presented by

We generate various starlike and convex univalent functions and polynomials through the location of their zeros and/or poles. It turns out that these starlike functions are extremal, in the whole class of univalent functions, for a functional which involves the Schwarzian derivative.

1 Introduction

Let $D = \{z \; : \; |z| < 1\}$ be the unit disc in the complex plane and $\mathcal{A}(D)$ be the set of analytic functions in D. We denote by $\mathcal{S}$ the class of univalent functions with the usual normalization,

$$\mathcal{S} = \{f \in \mathcal{A}(D) \; : \; f(0) = 0, f'(0) = 1, \; f(z) \neq f(\zeta) \text{ whenever } z \neq \zeta, z, \zeta \in D\}.$$

Alexander [1] was the first to introduce certain subclasses of univalent functions examining the geometric properties of the image $f(D)$ of D under f. The convex functions are those that map D onto a convex set. A function $w = f(z)$ is said to be starlike if, together with any of its points w, the image $f(D)$ contains the entire segment $\{tw \; : \; 0 \leq t \leq 1\}$. Thus we introduce the denotations

$$\begin{aligned} \mathcal{S}^* &= \{f \in \mathcal{S} \; : \; f \text{ is starlike in } D\}, \\ \mathcal{K} &= \{f \in \mathcal{S} \; : \; f \text{ is convex in } D\}. \end{aligned}$$

These classes obey very nice analytic characterizations, too. It can be proved that $f \in \mathcal{S}^*$ if and only if

$$(1.1) \qquad \Re\left(z\frac{f'(z)}{f(z)}\right) > 0 \quad \text{for every} \ \ z \in D,$$

and, similarly, $f \in \mathcal{K}$ if and only if

$$(1.2) \qquad \Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0 \quad \text{for every} \ \ z \in D.$$

We refer to a recent paper of Gluchoff and Hartmann [5] for a historical overview of Alexander's seminal paper and to Duren's book [4] for comprehensive information about univalent functions.

The characterizations (1.1) and (1.2) suggest that the distribution of the zeros and/or of poles of $f(z)$ outside the unit disc should provide information about the geometric properties of $f(D)$ and, vice versa, once we know the shape of $f(D)$, we could be able to locate the zeros and the poles of $f(z)$. This relation between the geometric properties and location of zeros and poles of univalent functions has been studied thoroughly. Recently Gluchoff and Hartmann [6] explored the idea in details, especially for the case of univalent polynomials. They also reviewed many of the contributions on this interplay.

In this note we generate various starlike and convex univalent polynomials and functions examining the behaviour of their zeros. We are interested in the particular case when there exist points on the boundary of the unit disc for which the above inequalities (1.1) and (1.2) reduce to equalities. In other words, all the starlike functions which will appear in the sequel satisfy

$$(1.3) \qquad \Re\left(z\frac{f'(z)}{f(z)}\right) = 0 \quad \text{for some} \ \ z \in \partial D,$$

and the convex functions satisfy

$$(1.4) \qquad \Re\left(1 + z\frac{f''(z)}{f'(z)}\right) = 0 \quad \text{for some} \ \ z \in \partial D.$$

It is seen that (1.1) is equivalent to the fact that the argument of $f(e^{i\theta})$ increases with $\theta \in [0, 2\pi)$. Then the vanishing of $\Re(zf'(z)/f(z))$ on the boundary is equivalent to the presence of cusps on the curve $f(e^{i\theta})$. Similarly, (1.2) is the analytic equivalent of the fact that the argument of the tangent line to the same curve increases. Thus $\Re(1 + zf''(z)/f'(z))$ vanishes at $z = e^{i\theta}$ only if the curvature of $f(e^{i\theta})$ vanishes, i.e., when this curve is not strictly convex.

Let $\mathcal{S}_n$ be the set of univalent polynomials of degree n from $\mathcal{S}$. For any such polynomial $p_n(z)$, denote by $z_1, z_2, \ldots, z_{n-1}$ its zeros that are located outside D, and by $\xi_1, \xi_2, \ldots, \xi_{n-1}$ its critical points (the zeros of $p'_n(z)$). We are particularly motivated by a result of Brannan [2] who proved that the only starlike polynomials of degree n with real coefficients and with all its critical points $\xi_1, \ldots, \xi_{n-1}$ on ∂D are $z \pm z^n/n$. Since there is no univalent polynomial with a critical point in D, Brannan's example shows that these polynomials are extremal in the sense that both the products $|\xi_1 \ldots \xi_{n-1}|$ and $|z_1 \ldots z_{n-1}|$ are the smallest possible. These facts follow also from a result of Dieudonné [3] who proved that the inequality $|a_n| \leq 1/n$ holds for the last coefficient of every $p_n \in \mathcal{S}_n$. It is easily seen then (see Theorem 1 below) that the polynomials $P_n(z) = z \pm z^n/n^2$ are convex and satisfy (1.4).

We shall construct various polynomials and functions that are starlike or convex and their zeros have a specific location. It turns out that most of our examples obey various extremal properties. In the last section we show that the starlike functions we construct maximize, within the class of univalent functions, a functional which involves the Schwarzian derivative.

2 Preliminaries

Let $f \in \mathcal{A}(D)$ and suppose that f can be represented in the form

$$(2.1) \quad f(z) = z\frac{p(z)}{q(z)} \quad \text{with} \quad p(z) = \prod_{k=1}^{\varpi_1}(1 - z/\zeta_k), \quad q(z) = \prod_{k=1}^{\varpi_2}(1 - z/\eta_k),$$

where ϖ_1 and ϖ_2 may be finite or infinity. A necessary condition that $f(z)$ is analytic and univalent in D is that all its zeros ζ_k and poles η_k lie outside D. Moreover, in this special case

$$\Re\left(z\frac{f'(z)}{f(z)}\right) = 1 + \Re\left(z\frac{p'(z)}{p(z)}\right) - \Re\left(z\frac{q'(z)}{q(z)}\right).$$

In the next section we shall consider the case when $q(z) \equiv 1$. In this case $f(z)$ reduces to a polynomial or to an entire function. In Section 3 we study the case $p(z) \equiv 1$, i.e. functions that possess only poles.

Using the properties of the Möbius map $w = z/(z - z_0)$ we readily obtain

Lemma 1 *Let $z_0 \notin D$.*
a) If $|z_0| > 1$, then

$$\frac{1}{1 - |z_0|} \leq \Re\left(\frac{z}{z - z_0}\right) \leq \frac{1}{1 + |z_0|} \quad \text{for every} \quad z \in \overline{D}.$$

Moreover, equality in the left-hand side inequality is attained if and only if $z = z_0/|z_0|$ and the right-hand side inequality reduces to equality only at $z = -z_0/|z_0|$.
b) If $|z_0| = 1$, then $\Re(z/(z - z_0)) \leq 1/2$ for every $z \in \overline{D}$ and the equality is attained if and only if $|z| = 1$, $z \neq z_0$.

Thus, if all the zeros and poles of a univalent function $f(z)$, defined by (2.1), are real and positive, then the minimal value of both $\Re(zp'(z)/p(z))$ and $\Re(zq'(z)/q(z))$ is attained at $z = 1$ and the maximal value of each of these functions is attained at $z = -1$. Using Lemma 1 we can draw a conclusion about the behaviour of transforms obtained from the Möbius one by rotation.

Corrollary 1 *Let $g(z) = 1 - z^m/r$, with $r > 1$. Then*

$$m/(1 - r) \leq \Re(zg'(z)/g(z)) \leq m/(1 + r)$$

where equality in the left-hand side inequality is attained only at $z = e^{2j\pi/m}$ and in the right-hand side one only at $z = e^{(2j+1)\pi/m}$, for $j = 0, 1, \ldots, m - 1$.

We have the following close relation between convex and starlike functions.

Theorem 1 *Let $F(z)$ be analytic in D. Then $F \in \mathcal{K}$ if and only if $zF'(z) \in \mathcal{S}^*$.*

This theorem shows that once a starlike function $f(z)$ is constructed, then one obtains immediately a corresponding convex function $F(z)$ by

$$(2.2) \qquad\qquad F(z) = \int_0^z \frac{f(\zeta)}{\zeta} d\zeta.$$

3   Starlike and convex polynomials and functions with zeros only

First we construct starlike and univalent polynomials. In what follows we denote by $\mathcal{S}_n$, $\mathcal{S}_n^*$ and $\mathcal{K}_n$ the set of univalent, starlike and convex algebraic polynomials of degree n.

We begin with the construction of some polynomials whose nonzero roots lie on the positive real line. If we consider the particular case when all these
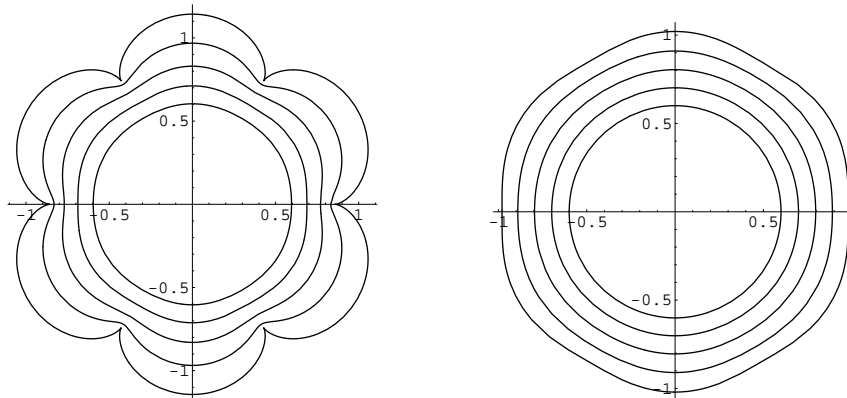
The image of ∂D and of the circumferences $|z| = \rho$, $\rho = 0.6, 0.7, 0.8, 0.9$, under the starlike polynomial $p_n(z) = z - z^n/n$ and the convex polynomial $P_n(z) = z - z^n/n^2$ for $n = 7$.

zeros coincide, the only starlike polynomial is $p_n(z) = z(1 - z/n)^{n-1}$ and its corresponding convex polynomial is $P_n(z) = 1 - (1 - z/n)^n$.

The second example is when the positive zeros of $p_n(z)$ are equidistant. If we set $\zeta_k = 1 + ck$, $k = 1, \ldots, n-1$, where c is a positive constant, then Lemma 1 a) and the requirement (1.3) yield $\zeta_k = 1 + H_{n-1}k$, for $k = 1, \ldots, n-1$, where $H_n = \sum_{j=1}^{n} 1/j$ is the harmonic number. Hence

$$p_n(z) = z \prod_{k=1}^{n-1} \left(1 - \frac{z}{1 + H_{n-1}k}\right),$$

and the corresponding convex polynomial is given by

$$P_n(z) = \int_0^z (p_n(\zeta)/\zeta)d\zeta$$

Next we consider the case when $f(z)$ is an entire function. Let us try to find such a function with zeros ζ_k, all being real and greater than one, and such that $f(z)$ is starlike, that is, satisfies (1.1) but we require the existence of a point $z \in \partial D$, such that (1.1) reduces to an equality at z. As we have already observed, in this case the smallest value of $\Re(zf'(z)/f(z))$ is attained at $z = 1$
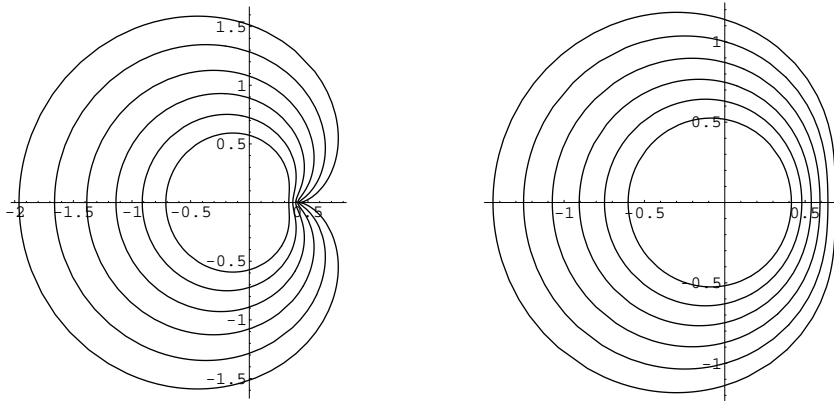
and then we must have

$$\sum_{k=1}^{\infty} \frac{1}{1 - \zeta_k} = -1.$$

If we choose $1 - \zeta_k = -ck^2$, with a positive constant c, then the sum $\sum_1^{\infty} 1/k^2 = \pi^2/6$ yields $\zeta_k = 1 + k^2\pi^2/6$. Now the representation

$$\sin(z) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2} \right)$$

implies that the starlike entire function with the desired zeros is

$$(3.1) \qquad\qquad f(z) = z \frac{\sqrt{6}}{\sinh\sqrt{6}} \frac{\sin\sqrt{6(z-1)}}{\sqrt{6(z-1)}}.$$

The image of ∂D and of the circumferences $|z| = \rho$, $\rho = 0.5, 0.6, 0.7, 0.8, 0.9$, under $f(z)$ and $F(z)$, defined by (3.1) and (3.2).

Applying the integration formula (2.2), we obtain the corresponding convex entire function

$$(3.2) \qquad\qquad F(z) = \frac{\sqrt{2/3}}{\sinh\sqrt{6}} \left(\cosh\sqrt{6} - \sin\sqrt{6(z-1)} \right).$$

Various further examples of starlike and convex functions, similar to (3.1) and (3.2) may be constructed. For example, if we set $1 - \zeta_k = -c(k - 1/2)^2$, $k = 1, \ldots$, then the sum $\sum_1^\infty 1/(k - 1/2)^2 = \pi^2/2$ yields $\zeta_k = 1 + (k - 1/2)^2 \pi^2/2$ and the corresponding starlike and convex functions are

$$(3.3) \qquad f(z) = z \frac{\cos \sqrt{2(z-1)}}{\cosh \sqrt{2}},$$

$$(3.4) \qquad F(z) = \frac{1}{\cosh \sqrt{2}} \left(\cos \sqrt{2(z-1)} + \sqrt{2(z-1)} \sin \sqrt{2(z-1)} \right.$$
$$\left. - \cosh \sqrt{2} + \sqrt{2} \sinh \sqrt{2} \right).$$

Similarly, if we set $\zeta_k = k^2$ for $k = 2, \ldots$, then we the requirement $\Re(f'(1)/f(1)) = 0$ yields $\zeta_1 = 5$. Thus we generate the following pair of starlike and convex functions:

$$(3.5) \qquad f(z) = z \frac{1 - z/5}{1 - z} \frac{\sin \pi \sqrt{z}}{\pi \sqrt{z}},$$

$$(3.6) \qquad F(z) = \frac{2}{5\pi^2} \left(1 - \cos \sqrt{\pi z} - 2\pi(Si(\pi(1 + \sqrt{z})) + \right.$$
$$\left. Si(\pi(1 - \sqrt{z})) - 2Si(\pi)) \right),$$

where $Si(z) = \int_0^z (\sin \zeta / \zeta) \, d\zeta$ is the sine integral function.

Next we shall construct an entire function with zeros located at the points of intersection on m rays with infinitely many circles. Then the function must be of the form $f(z) \equiv zp(z) = z \prod_{k=1}^\infty (1 - z^m/r_k)$, where $r_k > 1$. By Corollary 1 we conclude that

$$1 + \Re \left(z \frac{p'(z)}{p(z)} \right) \geq 1 + m \sum_{k=1}^\infty \frac{1}{1 - r_k}, \quad z \in \overline{D},$$

and the latter inequality reduces to equality if and only if $z = \exp(2ij\pi/m)$, for $j = 0, \ldots, m - 1$. Then we may choose the radii r_k in a similar way as in the previous examples. We omit this technical detail. The resulting starlike entire function is

$$(3.7) \qquad f(z) = z \frac{\sqrt{6/m}}{\sinh \sqrt{6/m}} \frac{\sin \sqrt{6(z^m - 1)/m}}{\sqrt{6(z^m - 1)/m}}.$$

4 Starlike and convex polynomials and functions with poles only

In this section we consider the case $p(z) \equiv 1$. In other words, our functions will posses only poles η_k. Recall that in this case

$$1 - \Re\left(z\frac{f'(z)}{f(z)}\right) = 1 - \Re\left(z\frac{q'(z)}{q(z)}\right)$$

and, when $\eta_k > 1$, the maximum of $\Re(z/(z - \eta_k))$ is attained at $z = -1$.

First we shall observe that it is impossible to transform the example where we generated the function (3.1). Indeed, if $1 + \eta_k = ck^2$, for $k = 1, 2, \ldots$, with a positive constant c, then the sum $\sum_1^\infty 1/k^2 = \pi^2/6$ yields $\eta_k = -1 + k^2\pi^2/6$. Then $\eta_1 \approx 0.645$ which means that $f(z)$ must possess a pole in D.

Consider now $\eta_k = k^2$, for $k = 2, \ldots$. In this case we can calculate η_1 using the equality

$$\frac{1}{1 + \eta_1} + \sum_{k=2}^\infty \frac{1}{1 + \eta_k} = 1,$$

and the result is

$$\eta_1 = \frac{2 - \pi \coth \pi}{-4 + \pi \coth \pi}.$$

Thus, we generate the starlike function

$$(4.1) \qquad f(z) = z\frac{\pi\sqrt{z}(1 - z)}{\left(1 - \frac{z}{\eta_1}\right)\sin \pi\sqrt{z}},$$

If we set $1 + \eta_k = c(k+1)$, for $k = 1, 2, 3$, then the requirement (1.3) yields $\eta_k = 13(k+1)/12 - 1$. So, the resulting pair of starlike and convex functions is

$$(4.2) \qquad f(z) \;=\; z\frac{1}{(1 - 6z/7)(1 - 4z/9)(1 - 3z/10)},$$

$$(4.3) \qquad F(z) \;=\; -\frac{630}{169}\log\left(\frac{81(3z - 10)(6z - 7)}{70(4z - 9)^2}\right).$$

Our next example concerns a function with infinitely many poles located at the intersection points of m rays and infinitely many circumferences. In other words, the function will be of the form

$$f(z) = z\frac{1}{q(z)} = z/\prod_{k=1}^\infty (1 - z^m/\epsilon_k),$$
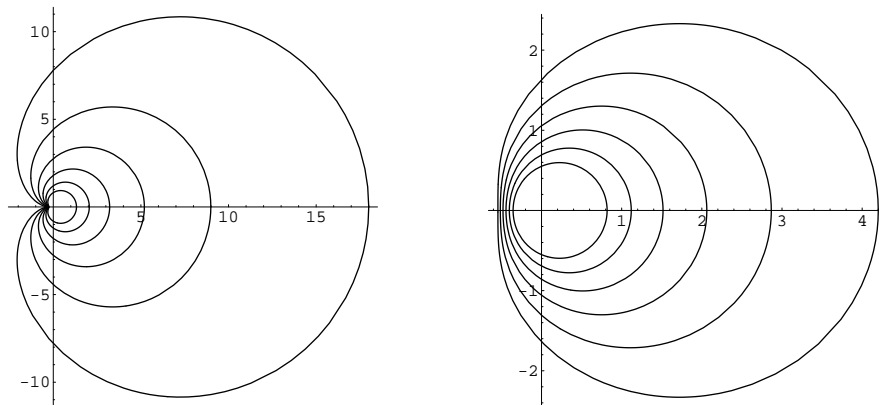
The image of ∂D and of the circumferences $|z| = \rho$, $\rho = 0.5, 0.6, 0.7, 0.8, 0.9$, under $f(z)$ and $F(z)$, defined by (4.2) and (4.3).

where $\epsilon_k > 1$. The conditions for starlikeness and Corollary 1 yield

$$1 - \Re\left(\frac{zq'(z)}{q(z)}\right) \geq 1 - m\sum_{k=1}^{\infty}\frac{1}{1+\epsilon_k}, \quad z \in \bar{D},$$

and the equality in the latter inequality is attained if only if $z = \exp(i(2j+1)\pi/m)$, $j = 0, ..., m-1$.

If we consider $(1+\epsilon_k)/m = c(k-1/2)^2$, condition (1.3) implies $\epsilon_k = m(k-1/2)^2\pi^2/2 - 1$ with $m > 1$. Thus we obtain the starlike function

(4.4)
$$f(z) = z\frac{\cos\sqrt{2/m}}{\cos\sqrt{2(z^m+1)/m}}$$

5   Starlike and convex polynomials and functions with zeros and poles

In this section, in accordance with the representation of f, neither $p \equiv 1$ nor $q \equiv 1$.

In the first example we consider real poles satisfying $\eta_k < -1$ and real

zeros with $\zeta_k > 1$. Then we have

$$1 + \Re\left(\frac{zp'(z)}{p(z)}\right) - \Re\left(\frac{zq'(z)}{q(z)}\right) \geq 1 + \sum_{k=1}^{\infty}\frac{1}{1-\zeta_k} - \sum_{k=1}^{\infty}\frac{1}{1-\eta_k}, \quad z \in \overline{D},$$

with equality being attained only for $z = 1$. Let us fix the zeros $\zeta_k = 1 + \pi^2 k^2/3$. Then an appropriate choice for the poles, in order (1.3) to be satisfied, is $\eta_k = 1 - (\pi^2 - 8)(k + 1/2)^2$, $k = 1, 2\ldots$, and we obtain the starlike function

$$(5.1)\, f(z) = z\frac{\cos(\pi/\sqrt{\pi^2 - 8})}{(\pi^2 - 12)\sinh\sqrt{3}}\frac{(4(z-3)+\pi^2)}{\sqrt{z-1}}\frac{\sin\sqrt{3(z-1)}}{\cosh\left(\pi\sqrt{(z-1)/(\pi^2-8)}\right)}.$$

Next we construct a starlike function with one pole on the unit circumference and infinitely many zeros. Setting $\eta_1 = -1$, from (1.3) we obtain $\zeta_k = \pi^2(k-1/2)^2 + 1$. This yields the starlike function

$$(5.2)\qquad\qquad\qquad f(z) = z\frac{1}{\cosh 1}\frac{\cos\sqrt{z-1}}{z+1}.$$

Another case is when both the zeros and the poles are located in m rays, but the poles are distributed on rotated rays in relation to the rays of the zeros. More precisely, we consider functions p and q of the form

$$(5.3)\qquad\qquad\qquad p(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z^m}{r_k}\right),$$

$$(5.4)\qquad\qquad\qquad q(z) = \prod_{k=1}^{\infty}\left(1 + \frac{z^m}{\epsilon_k}\right).$$

Then we have

$$1 + \Re\left(\frac{zp'(z)}{p(z)}\right) - \Re\left(\frac{zq'(z)}{q(z)}\right) \geq 1 + m\sum_{k=1}^{\infty}\frac{1}{1-r_k} - m\sum_{k=1}^{\infty}\frac{1}{1+\epsilon_k}, \quad z \in \overline{D},$$

with equality only for $z = \exp(i2j\pi/m)$. Then the right-hand side of the above inequality vanishes for $r_k = m\pi^2 k^2/3 + 1$ and $\epsilon_k = m\pi^2(k-1/2)^2 - 1$. Thus we obtain the starlike function

$$(5.5)\qquad f(z) = z\frac{\sqrt{3/m}\cos\sqrt{1/m}}{\sinh\sqrt{3/m}}\frac{\sin\sqrt{3(z^m-1)/m}}{\sqrt{3(z^m-1)/m}\cosh\sqrt{(z^m-1)/m}}.$$

Another interesting example is obtained if we require that the starlike function f possesses only one pole on the unit circumference and two real zeros

$r_1, r_2 > 1$, namely we consider $q(z) = (1 - z/e^{i\theta_1})$, with $0 \leq \theta_1 < 2\pi$, and $p(z) = (1 - z/r_1)(1 - z/r_2)$. Then

$$1 + \Re\left(\frac{zp'(z)}{p(z)}\right) - \Re\left(\frac{zq'(z)}{q(z)}\right) \geq \frac{1}{2} + \frac{1}{1 - r_1} + \frac{1}{1 - r_2}, \quad z \in \overline{D},$$

where, by Lemma 1, this inequality reduces to equality if and only if $z = 1$ provided $\theta_1 \neq 0$. We choose $\theta_1 = \pi/4$. If we set $1 - r_k = -ck$ for $k = 1, 2$, then the right-hand side vanishes only when $r_k = 3k + 1$. This leads to following pair of starlike and convex functions:

$$(5.6) \quad f(z) \quad = \quad z\frac{(1 - z/4)(1 - z/7)}{(1 - e^{-i\frac{\pi}{4}}z)}$$

$$(5.7) \quad F(z) \quad = \quad \frac{\left(22i - (27 + 29i)\sqrt{2}\right)\log(1 + e^{\frac{i3\pi}{4}}z) - \left(2i + e^{\frac{i\pi}{4}}(-22 + z)\right)z}{56}.$$
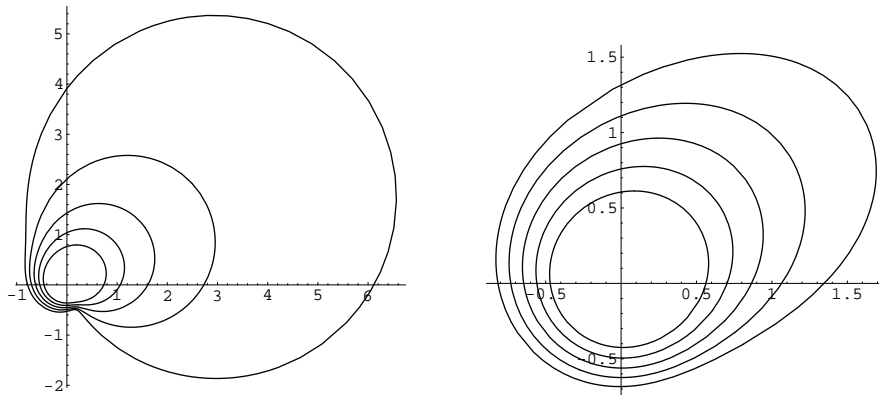
The image of the circumferences $|z| = \rho$, $\rho = 0.6, 0.7, 0.8, 0.9$, under $f(z)$ and $F(z)$, defined by (5.6) and (5.7).

Our last example concerns a function with the same pole on the unit circumference ∂D and $2m$ zeros located at the intersections of m rays and two circumferences with radii r_1 and r_2. Then we have again $q(z) = 1 - \exp(-i\pi/4)z$
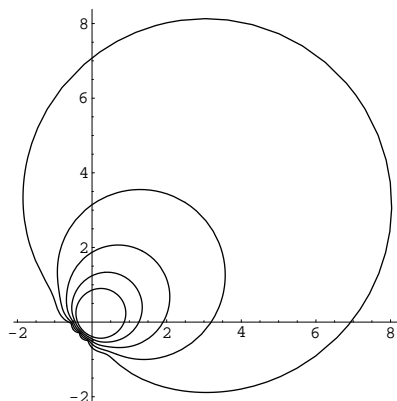
but now

$$p(z) = \prod_{k=1}^{2} \left(1 - \frac{z^m}{r_k} \right).$$

Then

$$1 + \Re \left(\frac{zp'(z)}{p(z)} \right) - \Re \left(\frac{zq'(z)}{q(z)} \right) \geq \frac{1}{2} + m \sum_{k=1}^{2} \frac{1}{1 - r_k}, \quad z \in \overline{D}.$$

Obviously inequality is attained at $z = 1$. If $(1 - r_k)/m = -ck^2$, then $r_k = 5k^2m/2 + 1$, for $k = 1, 2$, which yields the starlike function

$$(5.8) \qquad f(z) \;\; = \;\; z\frac{(1 - z^m/(1 + 5m/2))(1 - z^m/(1 + 10m))}{(1 - e^{-i\frac{\pi}{4}} z)}.$$

The image of the circumferences $|z| = \rho$, $\rho = 0.6, 0.7, 0.8, 0.9$, under $f(z)$ the starlike function (5.8) with $m = 5$.

6 Some extremal properties involving starlike functions

In this section we discuss an interesting extremal property involving the starlike functions that were generated in this paper. Recall that the Schwarzian derivative, or, simply, the schwarzian of a function w at z is denoted by $\{w, z\}$ and is

defined by

$$\{w, z\} = \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2.$$

In 1949 Zeev Nehari [7] proved the following interesting necessary condition for a function f to be univalent in terms of its schwarzian:

Theorem 2 *If $f(z) \in \mathcal{S}$, then*

(6.1) $(1 - |z|^2)^2|\{f, z\}| \leq 6,$

and 6 is the smallest possible constant to set on the right-hand side of (6.1).

That 6 is the best possible constant follows from the fact that for the Koebe function $k(z) = z/(1 - z)^2$

$$(1 - |z|^2)^2|\{k, z\}| = 6\frac{(1 - |z|^2)^2}{|1 - z^2|^2},$$

and the right-hand side of the latter equal 6 for every real z.

For Brannan's polynomials $p_n(z) = z - z^n/n$ we obtain

$$(6.2)(1 - |z|^2)^2|\{p_n, z\}| = \left|-\frac{(n-1)z^{n-3}(z^2-1)^2(2(n-2)+(n+1)z^{n-1})}{2(1-z^{n-1})}\right|$$

By L'Hospital rule the right-hand side of the latter converges to 6 when z tends to 1. In other words, we have

$$\lim_{z \to 1}(1 - |z|^2)^2|\{p_n, z\}| = 6.$$

Surprisingly, the limit 6 is attained for the majority of the starlike functions that we generated. We shall calculate the functional $(1 - |z|^2)^2|\{f, z\}|$ for some of them. Consider this functional for the starlike function (3.1). For real values of z we obtain

$$
\begin{aligned}
(1 - z^2)^2|\{f, z\}| \ = \ & \Big|3(z+1)^2 \Big(152 - 344z + 163z^2 + 90z^3 - 60z^4 + \\
& (40 - 88z + 65z^2 - 30z^3 + 12z^4)\cos(2\sqrt{6(z-1)} \\
& + 2\sqrt{6(z-1)}(4 + 2z - 11z^2 + 4z^3)\sin(2\sqrt{6(z-1)})\Big) / \\
& 16(\sqrt{6(z-1)}z\cos\sqrt{6(z-1)} + (z-2)\sin\sqrt{6(-1+z)})^2\Big|.
\end{aligned}
$$

Again, using L'Hospital rule, by lengthly but straightforward calculations we obtain

$$\lim_{z \to 1}(1 - z^2)^2|\{f, z\}| = 6.$$

Let us consider now the starlike function (4.2), restricting ourselves to real values of z. We obtain

$$(6.3) \qquad (1 - z^2)^2|\{f, z\}| = \left|\frac{6(z - 1)^2(16z^4 - 560z + 945)}{(8z^2 - 35z + 35)^2}\right|.$$

The maximum of this ratio, when $z \in [-1, 1]$, is attained at $z = -1$. Moreover, we have

$$\lim_{z \to -1}(1 - z^2)^2|\{f, z\}| = 6.$$

For the starlike function (5.2) the product (6.1) is

$$
\begin{aligned}
(1 - z^2)^2|\{f, z\}| \;=\; & \Big|\big((1 + z)^2(72 - 24z - 128z^2 + 61z^3 + 2z^4 + 5z^5 \\
& + (-24 + 48z - 22z^2 + 5z^3 + 4z^4 + z^5)\cos(2\sqrt{z - 1}) \\
& + 4\sqrt{z - 1}(30 - 41z + 15z^2 + 2z^3)\sin(2\sqrt{z - 1}))\big)/ \\
& (16(-2\sqrt{z - 1}\cos(2\sqrt{z - 1}) + z(1 + z)\sin(2\sqrt{z - 1}))^2)\Big|
\end{aligned}
$$

and again we obtain

$$\lim_{z \to 1}(1 - z^2)^2|\{f, z\}| = 6.$$

We have verified that the latter limit relation holds also for the starlike functions (3.5) and (3.7).

For the function (4.4) we obtain the limit relation

$$\lim(1 - |z|^2)^2|\{f, z\}| = 6 \quad \text{when} \quad z \to e^{i\pi/m} \quad \text{with} \quad |z| < 1.$$

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