APPROXIMATION THEORY:
A volume dedicated to Borislav Bojanov
(D. K. Dimitrov, G. Nikolov, and R. Uluchev, Eds.)
additional information (to be provided by the publisher)

Extended Cubature Formula of Turán type $(0, 2)$ for the Ball

Dimitar K. Dimitrov * and Guergana Petrova †

Dedicated with much admiration to Academician Borislav Bojanov on the occasion of his 60*th birthday*

We construct explicitly an extended cubature of Turán type $(0, 2)$ for the unit ball in $\mathbb{R}^n$. It is a formula for approximation of the integral over the ball by a linear combination of surface integrals over m concentric spheres, centered at the origin, of the integrand itself and its Laplacian. This extended cubature integrates exactly all $2m+1$-harmonic functions and hence all polynomials in n variables of degree $4m+1$.

1. Introduction

Let $\mathbb{R}^n$ be the real n-dimensional Euclidean space. The points in $\mathbb{R}^n$ are denoted by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $|\mathbf{x}|$ is the Euclidean norm of $\mathbf{x}$. Given $r > 0$, let $B(r) = \{\mathbf{x} : |\mathbf{x}| < r\}$ and $S(r) = \{\mathbf{x} : |\mathbf{x}| = r\}$ be the ball and the hypersphere with center 0 and radius r in $\mathbb{R}^n$, respectively. If $r = 1$, we omit the r in the notation and simply write B and S. We denote by $d\mathbf{x}$ the Lebesgue measure in $\mathbb{R}^n$ and by $d\sigma$ the $(n-1)$-dimensional surface measure on $S(r)$. Recall that the area of the unit sphere S in $\mathbb{R}^n$ is

$$\gamma_n = \frac{n\pi^{n/2}}{\Gamma(n/2+1)}.$$

*Research supported by the Brazilian Science Foundations FAPESP under Grant No. 2003/01874-2 and CNPq under Grant No. 300645/95-3.
†Research supported by the National Science Foundation grant DMS 0296020.

If $\mathcal{D}$ is a simply connected domain in $\mathbb{R}^n$, the function u, defined on $\mathcal{D}$, is called a polyharmonic function of order p (or p-harmonic function) if $u \in C^{2p-1}(\bar{\mathcal{D}}) \cap C^{2p}(\mathcal{D})$ and it satisfies the equation

$$\Delta^p u(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathcal{D}, \quad \text{where} \quad \Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad \Delta^p := \Delta\Delta^{p-1}.$$

In the case of $p = 1$ ($p = 2$), u is called harmonic (biharmonic). The set of all p-harmonic functions on $\mathcal{D}$ is denoted by $H^p(\mathcal{D})$. We refer the reader to [1, 10] for detailed information about polyharmonic functions. In what follows, $\pi_m(\mathbb{R}^n)$ denotes the linear space of all real algebraic polynomials of n variables whose total degree does not exceed m. In particular, π_m is the set of the univariate algebraic polynomials of degree at most m. It is clear that

$$\pi_{2p-1}(\mathbb{R}^n) \subset H^p(B). \tag{1.1}$$

It can be easily seen that the Gaussian mean value property

$$\int_B u(\mathbf{x})\ d\mathbf{x} \ = \ \frac{\pi^{n/2}}{\Gamma(n/2+1)} \cdot u(0),$$

the corollary of the first Green formula

$$\int_B u(\mathbf{x})\ d\mathbf{x} \ = \ \frac{1}{n} \cdot \int_S u(x)\ d\sigma,$$

(both valid for harmonic functions), and the Pizzetti's mean value formula

$$\int_B u(\mathbf{x})\ d\mathbf{x} \ = \ \sum_{j=0}^{m-1} \frac{\pi^{n/2}}{2^{2j} j! \Gamma(n/2+j+1)} \cdot \Delta^j u(0),$$

(which holds for all m-harmonic functions u), are natural extensions of the rectangular quadrature

$$\int_{-1}^{1} f(x)\ dx \ \approx \ 2 \cdot f(0),$$

the trapezoid quadrature rule

$$\int_{-1}^{1} f(x)\ dx \ \approx \ f(-1) + f(1),$$

(both precise for univariate polynomials of first degree), and the Taylor quadrature formula

$$\int_{-1}^{1} f(x)\ dx \ \approx \ \sum_{j=0}^{m-1} \frac{2}{(2j+1)!} \cdot f^{(2j)}(0),$$

(exact for all algebraic polynomials of degree $2m-1$), respectively. The idea of extending other univariate quadrature formulae for the interval $(-1,1)$ to cubature for approximating the integral over the unit ball, $I(u)=\int_B u(\mathbf{x})\,d\mathbf{x}$, by linear combinations of surface integrals of the integrand and its differential operators, was proposed in [7], where each hypersphere in $\mathbb{R}^n$ was considered as an extension of a pair of symmetric nodes in $\mathbb{R}$. Such cubature rules are called *extended cubature formulae* and are denoted by $Q(u)$. Recently, a substantial progress in constructing extended cubature formulae has been achieved. We refer the reader to Bojanov's nice survey [4] for an overview of the latest results in this area. An extended formula is said to have *polyharmonic order of precision m*, $PHOP(Q)=m$, if $I(u)=Q(u)$ for all $u\in H^m(B)$ and there exists a function $u\in H^{m+1}(B)$, for which $I(u)\neq Q(u)$. Note that, by (1.1), $Q(u)$ is precise for $\pi_{2p-1}(\mathbb{R}^n)$ if its $PHOP$ is at least p.

An interesting question concerning quadrature formulae of high algebraic degree of precision was considered by Paul Turán in [13, Problem XXXIII], where he posed the problem of finding a $(0,2)$ quadrature formula, namely a quadrature of the form

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=1}^{m} \left(a_k f(x_k) + c_k f''(x_k)\right),$$

whose algebraic degree of precision is at least $2m$. Recall that there is no quadrature of the form

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=1}^{m} \left(a_k f(x_k) + b_k f'(x_k)\right),$$

that is exact for all the univariate polynomials of degree $2m$. Turán's problem was solved in [6], where it was proved that the formula

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{m} \frac{2}{m(m+3)} \left(\frac{f(x_i)}{P_{m+1}^2(x_i)} + \frac{(1-x_i^2)\ f''(x_i)}{(m+1)(m+2)P_{m+1}^2(x_i)} \right), \quad (1.2)$$

has algebraic degree of precision $2m+1$. Here $P_m(x)$ are the Legendre polynomials, normalized by $P_m(1)=1$, and the nodes $x_i,\ i=1,\dots,m$, coincide with the zeros of $P'_{m+1}(x)$.

A natural question is then whether one can extend this result and construct an extended cubature formula of Turán type (0,2) for the ball, i.e. a cubature of the form

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx \sum_{j=1}^{m} \left(A_j \int_{S(r_j)} u(\xi)\,d\sigma(\xi) + C_j \int_{S(r_j)} \Delta u(\xi)\,d\sigma(\xi) \right),$$

that is exact for all polyharmonic functions of order $2m+1$. In this paper, we provide an affirmative answer to this question and derive the analog of quadrature (1.2). Before going further, let us introduce some notation:

let $P_m^{(1,n/2-1)}(x)$ be the Jacobi polynomials that are orthogonal on $(-1,1)$ with respect to the weight function $(1-x)(1+x)^{n/2-1}$ and normalized by $P_m^{(1,n/2-1)}(1) = m+1$. In what follows the zeros of $P_m^{(1,n/2-1)}(x)$ will be denoted by $x_1,\ldots,x_m$. Our main result is the following theorem.

Theorem 1.1. *For every $m \in \mathbb{N}$ the extended cubature formula*

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx \sum_{k=1}^{m} \frac{2^{(n-3)/2}(4m+n+4)^2}{m(2m+n+2)^3}\,\frac{(1+x_k)^{(3-n)/2}}{[P_{m+1}^{(1,n/2-1)}(x_k)]^2} \times \left(\int_{S(t_k)} u(\xi)\,d\sigma(\xi) + \frac{1-x_k}{4(m+1)(2m+n)}\int_{S(t_k)} \Delta u(\xi)\,d\sigma(\xi)\right), \tag{1.3}$$

where the radii $t_1,\ldots,t_m$ are given by $t_k = \sqrt{(1+x_k)/2}$, is precise for every $u \in H^{2m+1}(B)$.

2. Lobatto extended cubature for the ball in $\mathbb{R}^n$

In this section, we investigate extended formulae of the form

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx \widetilde{A}\int_{S(1)} u(\xi)\,d\sigma(\xi) + \sum_{k=1}^{m} \widetilde{A}_k \int_{S(t_k)} u(\xi)\,d\sigma(\xi), \tag{2.1}$$

that have maximal possible polyharmonic order of precision. We call such a formula a Lobatto extended cubature formula.

Theorem 2.1. *There is a unique cubature of the form (2.1) with polyharmonic order of precision $2m+1$. Its nodes are $t_k = \sqrt{(1+x_k)/2}$ and its coefficients are given by*

$$\widetilde{A} = \frac{1}{(m+1)(2m+n)}, \tag{2.2}$$

$$\widetilde{A}_k = \frac{2^{(n-3)/2}(4m+n+4)^2}{(m+1)(2m+n)(2m+n+2)^2}\,\frac{(1+x_k)^{(3-n)/2}}{[P_{m+1}^{(1,n/2-1)}(x_k)]^2}. \tag{2.3}$$

Moreover, there is no such a cubature with $PHOP = 2m+2$.

The main tool in the proof of the theorem is the following relation between quadrature and extended cubature rules (see [5, Lemma 3]):

Lemma 2.1. *Assume that $\mu(t)$ is a fixed weight function on $[0,1]$. Let $0 \le t_1 < \cdots < t_N \le 1$. The extended cubature formula*

$$\int_B \mu(|\mathbf{x}|)u(\mathbf{x})d\mathbf{x} \approx \sum_{k=1}^{N} a_k\,\frac{1}{\gamma_n t_k^{n-1}}\int_{S(t_k)} u(\xi)\,d\sigma(\xi)$$

is exact for every polyharmonic function $u \in H^m(B)$ if and only if the quadrature formula

$$\gamma_n \int_0^1 \mu(t) t^{n-1} P(t^2)\, dt \approx \sum_{k=1}^N a_k P(t_k^2)$$

is exact for every algebraic polynomial $P \in \pi_{m-1}$.

We shall need also the following technical result concerning a specific Radau quadrature formula.

Lemma 2.2. *The quadrature formula*

$$\int_{-1}^1 (1+x)^{n/2-1} f(x)\, dx \approx \mu f(1) + \sum_{k=1}^m \mu_k f(x_k), \tag{2.4}$$

where

$$\mu = \frac{2^{n/2+1}}{(m+1)(2m+n)}, \tag{2.5}$$

$$\mu_k = \frac{2^{n/2}(4m+n+4)^2}{(m+1)(2m+n)(2m+n+2)^2} \cdot \frac{1+x_k}{[P_{m+1}^{(1,n/2-1)}(x_k)]^2}, \tag{2.6}$$

is the only quadrature formula of this form that has algebraic degree of precision $2m$.

The existence and uniqueness of the quadrature rule of the above form with highest algebraic degree of precision is a classical result. It is called a Radau quadrature formula. The weights μ and μ_k have been calculated in the general case of Jacobi weight. We refer the reader to a recent paper of Gautschi [9] for the details. However, we obtained formulae (2.5) and (2.6) independently. It turns out that our method is different from Gautschi's one and, in some sense, more straightforward, so we are tempted to present it briefly. Let us recall first that the generalized hypergeometric series is defined by

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{z^j}{j!},$$

where $(a)_j$ is the Pochhammer symbol, defined by $(a)_0 = 1$ and $(a)_j = a(a+1)\cdots(a+j-1) = \Gamma(a+j)/\Gamma(a)$ for $j = 1, 2, \ldots$. The Gaussian hypergeometric function ${}_2F_1$ will be succinctly denoted by F.

In order to derive the coefficient μ, we apply (2.4) to the hypergeometric polynomial

$$Q(x) = F(-m, m+n/2+1; 2; (1-x)/2) = \sum_{j=0}^m \frac{(-m)_j (m+n/2+1)_j}{(2)_j} \frac{(1-x)^j}{2^j j!}.$$

Since the Jacobi polynomials are hypergeometric polynomials,

$$P_m^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_m}{m!} F(-m, m+\alpha+\beta+1; \alpha+1; (1-x)/2),$$

then $Q(x)$ is a constant multiple of $P_m^{(1,n/2-1)}(x)$. Since obviously $Q(1) = 1$, then the application of (2.4) to $Q(x)$ immediately yields

$$\mu = \int_{-1}^{1} (1+x)^{n/2-1} Q(x)\, dx.$$

Performing the change of variables $x + 1 = 2t$ in this integral and using the explicit representation of the Beta function in terms of Gamma functions, we obtain

$$\begin{aligned}
\mu &= 2^{n/2} \int_0^1 t^{n/2-1} F(-m, m+n/2+1; 2; 1-t)\, dt \\
&= 2^{n/2} \sum_{j=0}^{m} \frac{(-m)_j (m+n/2+1)_j}{(2)_j j!} \int_0^1 t^{n/2-1} (1-t)^j\, dt \\
&= 2^{n/2} \sum_{j=0}^{m} \frac{(-m)_j (m+n/2+1)_j}{(2)_j} \frac{\Gamma(n/2)}{\Gamma(n/2+j+1)} \\
&= \frac{2^{n/2+1}}{n} \sum_{j=0}^{m} \frac{(-m)_j (m+n/2+1)_j}{(2)_j (n/2+1)_j} \\
&= \frac{2^{n/2+1}}{n}\, {}_3F_2(-m, m+n/2+1, 1; 2, n/2+1; 1).
\end{aligned}$$

Applying Saalschütz's formula (see [8, p. 66, Eq. (30)] or [2, p. 9, Eq. (1)])

$${}_3F_2(-m, a, b; c, 1+a+b-c-m; 1) = \frac{(c-a)_m (c-b)_m}{(c)_m (c-a-b)_m}$$

with $a = m + n/2 + 1, b = 1$ and $c = 2$, we finally obtain the explicit formula (2.5) for μ.

Now, we continue with the calculation of the coefficients μ_k. Observe that since (2.4) has algebraic degree of precision $2m$, then whatever the polynomial $f \in \pi_{2m-1}$ is, it integrates exactly the polynomial $(1-x)f(x)$, that is

$$\int_{-1}^{1} (1+x)^{n/2-1} (1-x) f(x)\, dx \approx \sum_{k=1}^{m} \mu_k (1-x_k) f(x_k).$$

Therefore this is the unique Gaussian quadrature on $(-1,1)$ with weight function $(1-x)(1+x)^{n/2-1}$. The nodes of this Gaussian quadrature are the zeroes of $P_m^{(1,n/2-1)}$ and the coefficients λ_k are given by (see [12, p. 352, Eq.(15.3.1)])

$$\lambda_k = 2^{n/2+2} \frac{m+1}{2m+n} (1-x_k^2)^{-1} \left\{ P_m^{(1,n/2-1)\prime}(x_k) \right\}^{-2}.$$

Since $\lambda_k = \mu_k(1-x_k)$, we obtain

$$\mu_k = 2^{n/2+2}\frac{m+1}{2m+n}(1+x_k)\left\{(1-x_k^2)P_m^{(1,n/2-1)\prime}(x_k)\right\}^{-2}.$$

It remains to employ the second formula (4.5.7) from Szegő's book [12],

$$(1-x_k^2)P_m^{(1,n/2-1)\prime}(x_k) = -\frac{2(m+1)(2m+n+2)}{4m+n+4}P_{m+1}^{(1,n/2-1)}(x_k),$$

to derive (2.6).

Proof of Theorem 2.1: If $|\mathbf{x}| = t_j$ is the equation of the sphere $S(t_j)$, then it is clear that (2.1) is not exact for

$$\omega(\mathbf{x}) := (|\mathbf{x}|^2-1)\prod_{j=1}^{m}(|\mathbf{x}|^2-t_j^2)^2 \in \pi_{4m+2}(\mathbb{R}^n) \subset H^{2m+2}(B),$$

and therefore

$$PHOP(2.1) \leq 2m+1.$$

In order to prove the existence and the uniqueness, we employ Lemma 2.1 which implies that an extended cubature (2.1) with $PHOP = 2m+1$ exists and it is unique if and only if there exists a unique quadrature of the form

$$\int_0^1 t^{n-1}P(t^2)\,dt \approx BP(1) + \sum_{k=1}^{m} B_k P(t_k^2), \tag{2.7}$$

which is precise for every $P \in \pi_{2m}$ and whose coefficients are related to those of (2.1) by $B = A$ and $B_k = \widetilde{A}_k/t_k^{n-1}$. Simple change of variables $1+x = 2t^2$ shows that the latter fact is equivalent to the existence and uniqueness of the Radau quadrature (2.4) where the nodes and the coefficients of (2.7) and (2.4) are related by $1+x_k = 2t_k^2$, $\mu = 2^{n/2+1}B$ and $\mu_k = 2^{n/2+1}B_k$. Summarizing the relations between the formulae (2.1), (2.7) and (2.4), and having in mind Lemma 2.2, we conclude that there exists a unique extended Lobatto cubature formula (2.1) with $PHOP = 2m+1$. Moreover the radii t_k and the coefficients $\widetilde{A}$ and $\widetilde{A}_k$ are given by

$$\begin{aligned} t_k &= \sqrt{(1+x_k)/2}, \quad \text{with } P_m^{(1,n/2-1)}(x_k) = 0,\ k = 1,\dots,m,\\ \widetilde{A} &= \frac{1}{(m+1)(2m+n)},\\ \widetilde{A}_k &= \frac{2^{-3/2}}{(1+x_k)^{(n-1)/2}}\mu_k, \end{aligned}$$

and the latter yields the explicit representation (2.3) of $\widetilde{A}_k$. This completes the proof of Theorem 2.1.

It is clear from Almansi's expansion (see [1] and [3, 11] for recent extensions), that if $u \in H^{2m}(B)$, then $(1-|\mathbf{x}|^2)u \in H^{2m+1}(B)$ and application of (2.1) to the latter function gives

$$\int_B (1-|\mathbf{x}|^2)u(\mathbf{x})\,d\mathbf{x} = \sum_{k=1}^{m} \widetilde{A}_k(1-t_k^2)\int_{S(t_k)} u(\xi)\,d\sigma(\xi), \tag{2.8}$$

which holds for every $u \in H^{2m}(B)$.

3. Extended cubature of Turan type $(0,2)$ for the ball in $\mathbb{R}^n$

In this section, we follow the approach from [6] and derive in explicit form the extended cubature of Turán type $(0,2)$.

Proof of Theorem 1.1: The first Green formula [1, p. 10] yields

$$\int_B u(\mathbf{x})\,d\mathbf{x} = \frac{1}{n}\int_{S(1)} u(\xi)\,d\sigma(\xi) - \frac{1}{2n}\int_B (1-|\mathbf{x}|^2)\Delta u(\mathbf{x})\,d\mathbf{x},$$

which holds for every $u \in C^2(B)\bigcap C(\overline{B})$. In particular, if $u \in H^{2m+1}(B)$, then $\Delta u \in H^{2m}(B)$ and application of the Gaussian extended cubature formula (2.8) to Δu gives

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx \frac{1}{n}\int_{S(1)} u(\xi)\,d\sigma(\xi) - \frac{1}{2n}\sum_{k=1}^{m} \widetilde{A}_k(1-t_k^2)\int_{S(t_k)} \Delta u(\xi)\,d\sigma(\xi) \tag{3.1}$$

that is exact for every $u \in H^{2m+1}(B)$. Now, we multiply (2.1) by γ and (3.1) by δ, where

$$\gamma = \frac{(m+1)(2m+n)}{m(2m+n+2)}, \qquad \delta = -\frac{n}{m(2m+n+2)},$$

and add the results to obtain (1.3). Since both (2.1) and (3.1) integrate exactly the functions in $H^{2m+1}(B)$, the extended Turán type $(0,2)$ cubature (1.3) is exact for all functions in $H^{2m+1}(B)$.

Remark 1. *In the case* $m=1$*, the Turán type cubature* (1.3) *is precise for the* 3*-harmonic functions and reduces to*

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx \frac{1}{n}\left(\frac{n+4}{n}\right)^{\frac{n-1}{2}}\left(\int_{S(\rho)} u\,d\sigma + \frac{1}{(n+2)(n+4)}\int_{S(\rho)} \Delta u\,d\sigma\right), \tag{3.2}$$

where

$$\rho := \left(\frac{n}{n+4}\right)^{1/2}.$$

This formula was obtained for the first time in [4], using other technique.

Remark 2. *For $n = 1$ the Turán $(0, 2)$ extended cubature (1.3) reduces to the quadrature (1.2), where the number of nodes is $2m$. The same is true for the Lobatto cubature (2.1) and for the Gaussian extended cubature (2.8). This occurs because the univariate natural analog of a surface integral is a pair of symmetric nodes. It is interesting to observe also that, if we set $n = 1$ in (3.2), we obtain exactly the Turán $(0, 2)$ quadrature (1.2) with two nodes.*

References

[1] N. Aronszajn, T. Greese and L. Lipkin, "Polyharmonic functions", Clarendon Press, Oxford, 1983.

[2] W. N. Bailey, "Generalized Hypergeometric Series", Cambridge University Press, Cambridge, 1935.

[3] B. Bojanov, An extension of the Pizzetti formula for polyharmonic functions, *Acta Math. Hungar.* **91** (2001) 99–113.

[4] B. Bojanov, Cubature formulae for polyharmonic functions, *in* "Recent progress in multivariate approximation" (Witten-Bommerholz, 2000), Internat. Ser. Numer. Math., Vol. 137, pp. 49–74, Birkhäuser, Basel, 2001.

[5] B. Bojanov and D. K. Dimitrov, Gaussian extended cubature formulae for polyharmonic functions, *Math. Comp.* **70** (2000) 671–683.

[6] D. K. Dimitrov, On a problem of Turán: $(0, 2)$ quadrature formula with a high algebraic degree of precision, *Aequationes Math.* **41** (1991), 168–171.

[7] D. K. Dimitrov, Integration of polyharmonic functions, *Math. Comp.* **65** (1996) 1269–1281.

[8] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions, Vol I, McGraw-Hill, New York, 1953.

[9] W. Gautschi, Gauss-Radau formulae for Jacobi and Laguerre weight functions, *Math. Comput. Simulation* **54** (2000) 403–412.

[10] M. Nicolescu, " Les Fonctions Polyharmoniques", Hermann, Paris, 1936.

[11] G. Petrova, Uniqueness of the Gaussian extended cubature for polyharmonic functions, *East J. Approx.* **9**(3) (2003), 269–275.

[12] G. Szegő, "Orthogonal Polynomials", 4th ed., Amer. Math. Soc. Collooq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1975.

[13] P. Turán, On some open problems of approximation theory, *J. Approx. Theory* **29** (1980), 23–85.

Dimitar K. Dimitrov

Departamento de Ciências de Computação e Estatística
Instituto de Biociências Letras e Ciências Exatas

Universidade Estadual Paulista
15054-000 São José do Rio Preto, SP
Brazil
E-mail: dimitrov@dcce.ibilce.unesp.br

Guergana Petrova
Department of Mathematics
Texas A&M University
College Station, TX 77843
USA;
E-mail: gpetrova@math.tamu.edu